Constructing bounded pseudoconvex domains of finite D'Angelo type with prescribed weak loci

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Abstract

We construct smoothly bounded pseudoconvex domains in $\mathbb{C}^n$, convex in certain cases, whose weakly pseudoconvex loci realize prescribed closed sets. The construction is based on Whitney extension, local smooth plurisubharmonic solutions of the complex Monge–Ampère equation, and a regularized maximum function used to patch local models with exterior barriers. We also obtain explicit upper bounds for the D'Angelo type along the prescribed weakly pseudoconvex locus. Finally, we formulate two conjectures relating finite D'Angelo type, nonflatness of the Levi determinant, and the size of the weakly pseudoconvex locus.

1 Introduction

A fundamental problem in several complex variables and complex geometry is to understand the geometry and analysis of weakly pseudoconvex boundary points of pseudoconvex domains. One of the central invariants in this direction is D'Angelo type, which measures the maximal order of contact between the boundary and germs of holomorphic curves passing through a given boundary point.

For smooth hypersurfaces locally defined by real-analytic functions, the structure of points of infinite D'Angelo type is rigidly constrained. A landmark theorem of D'Angelo [2] states that the D'Angelo type at a point of such a hypersurface is infinite if and only if the hypersurface contains a germ of a nonconstant holomorphic curve through that point. On the other hand, according to a global theorem of Diederich–Fornæss [5], every compact real-analytic variety in $\mathbb{C}^n$ contains no such germ. Combining these two facts, one obtains that every smoothly bounded pseudoconvex domain with real-analytic boundary is of finite D'Angelo type at every boundary point.

While real-analyticity forces infinite-type points to arise from complex analytic varieties, this characterization fails for general smooth hypersurfaces. Nevertheless, D'Angelo proved in [2] that the set of points of finite type is open on the hypersurfaces. In particular, the condition that the D'Angelo type is equal to 2 corresponds to Levi non-degeneracy and hence is open. For pseudoconvex domains, this is precisely strict pseudoconvexity. The weakly pseudoconvex locus is the set of boundary points at which strict pseudoconvexity

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fails. This locus is important not only geometrically but also analytically: in the $\bar{\partial}$-Neumann problem, the availability of local subelliptic estimates depends sensitively on its size, flatness, and structure. The role of the weakly pseudoconvex locus is also evident in the celebrated Diederich–Fornæss worm domain [4], which has served as a fundamental example permeating the field of several complex variables and showing that smooth pseudoconvexity alone does not guarantee the expected global analytic behavior. Its weakly pseudoconvex boundary structure is closely tied to phenomena such as the failure of Stein neighborhood bases and the irregularity of canonical operators.

In a related forthcoming work, the second author, jointly with Shao, Shi and Wu [16], has shown that, given a smoothly bounded real-analytic pseudoconvex domain Ω, there exists a smooth perturbation Ω_ϵ of Ω whose weakly pseudoconvex locus coincides with that of Ω. Moreover, Ω_ϵ is strictly pseudoconvex and real-analytic away from this locus. At each point of this locus, the boundary of Ω_ϵ fails to be real-analytic, while the corresponding D'Angelo type coincides with that of the original domain.

Unlike the rigid analytic structure of weakly pseudoconvex loci in the real-analytic category, smooth pseudoconvex boundaries may exhibit substantially greater flexibility. The purpose of this paper is to address a complementary realization phenomenon: in the smooth category, weakly pseudoconvex loci can be prescribed a priori with considerable freedom, including sets that need not arise from real-analytic geometry, while retaining finite type control. Thus, even under a uniform finite-type bound, weak loci may have exotic non-analytic structures. Throughout the paper, we write points in $\mathbb{C}^n$ as $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

The first theorem constructs a smooth convex domain of finite type in $\mathbb{C}^2$ whose weakly pseudoconvex locus has the structure of a family of tree-like sets in the $\operatorname{Im} w$ (i.e., the Reeb) direction, rooted at the points of a prescribed compact set E in the z (complex tangential) direction.

Theorem 1.1. *Let $E \subset \mathbb{R} \subset \mathbb{C}$ be an arbitrary compact set. For every integer $m \geq 2$, there exist a smoothly bounded convex domain $\Omega \subset \mathbb{C}^2$ and a neighborhood $\mathcal{U}$ of the set $(E + i0) \times (\mathbb{R} + i0)$ in $\mathbb{C}^2$, such that the weakly pseudoconvex locus of Ω is precisely*

$$W = \{(z, w) \in b\Omega : \operatorname{Re} z \in E,\ \operatorname{Im} z = 0\} \cap \overline{\mathcal{U}}.$$

Moreover, the D'Angelo type is equal to $2m$ at every point in W.

By adding a controlling term in the $\operatorname{Im} w$ direction to the defining equation, we shall eliminate the fibers in that direction and obtain a smoothly bounded convex domain whose weakly pseudoconvex locus is precisely the prescribed compact set.

Theorem 1.2. *Let $E \subset \mathbb{R} \subset \mathbb{C}$ be an arbitrary compact set. For every integer $m \geq 2$, there exists a smoothly bounded convex domain $\Omega \subset \mathbb{C}^2$, such that the weakly pseudoconvex locus of Ω is precisely*

$$W = \{(z, w) \in b\Omega : \operatorname{Re} z \in E,\ \operatorname{Im} z = 0 \text{ and } \operatorname{Im} w = 0\} \cong_{\textit{diff}} E.$$

Moreover, the D'Angelo type is equal to $2m$ at every point in W.

A surprising special case occurs when E is a Cantor set, which is totally disconnected but may have either zero or positive Lebesgue measure. More generally, depending on the construction, a Cantor set in $\mathbb{R}$ may have any prescribed Hausdorff dimension between 0 and 1. In this setting, the construction in Theorem 1.2 can realize any such Cantor set as the weakly pseudoconvex locus, while Theorem 1.1 produces what may be regarded intuitively as a "Cantor forest" as the weakly pseudoconvex locus; see Figure 1.

In higher dimensions, we have constructed a similar result as in Theorem 1.1, although in this setting the construction produces pseudoconvex domains.

Theorem 1.3. *Let $n \geq 3$ and $E \subset \mathbb{R}^{n-1} \subset \mathbb{C}^{n-1}$ be an arbitrary closed set. For every integer $m \geq 2$ and every point $x_0 \in E$, there exist a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and a neighborhood $\mathcal{U}$ of the real line $\{x_0 + i0\} \times (\mathbb{R} + i0)$ in $\mathbb{C}^n$, such that the weakly pseudoconvex locus of Ω is precisely*

$$W = \{(z, w) \in b\Omega : \operatorname{Re} z \in E,\ \operatorname{Im} z = 0\} \cap \overline{\mathcal{U}}.$$

Moreover, the D'Angelo type is less than or equal to $2m + 2$ at every point in W.

If the prescribed closed set E is of real dimension $n - 2$, then the construction can be altered to produce convex domains, as given below.

Theorem 1.4. *Let $n \geq 3$ and $E \subset \mathbb{R}^{n-2} \subset \mathbb{C}^{n-2}$ be an arbitrary closed set. For every integer $m \geq 2$ and every point $\hat{x}_0 \in E$, there exist a smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$ and a neighborhood $\mathcal{U}$ of the real line $\{(\hat{x}_0, 0) + i0\} \times (\mathbb{R} + i0)$ in $\mathbb{C}^n$, such that the weakly pseudoconvex locus of Ω is precisely*

$$W = \{(z, w) \in b\Omega : \operatorname{Re} z \in E \times \{0\},\ \operatorname{Im} z = 0\} \cap \overline{\mathcal{U}}.$$

Moreover, the D'Angelo type is less than or equal to $2m$ at every point in W.

Our construction is based on three main ingredients. First, we make use of the Whitney extension theorem to construct hypersurface models whose weak loci agree with the prescribed closed sets. When $n = 2$, the construction can be carried out using elementary subharmonic potentials, since the weak pseudoconvexity condition corresponds to a linear Poisson equation in this dimension. The hypersurfaces obtained by this construction are, however, initially unbounded. When $n \geq 3$, the weak pseudoconvexity condition is reduced to a fully nonlinear partial differential equation. We instead use local solvability results for the degenerate elliptic Monge–Ampère equation, due to Hong and Zuily [9] and Kallel-Jallouli [10], to obtain smooth convex/plurisubharmonic potentials and hence the desired local hypersurface models. Secondly, in order to pass from unbounded or local hypersurfaces to bounded global hypersurfaces without changing the prescribed weak loci, we use a symmetric smooth regularized maximum function. This function is obtained by mollifying the absolute value function and is designed to preserve the monotonicity and positive semi-definiteness properties. Finally, we introduce an anisotropic ellipsoidal global barrier, which, together with the regularized maximum function, allows us to perform the patching construction and yield smoothly bounded global domains, while preserving the weakly pseudoconvex loci and keeping the D'Angelo type under the prescribed upper bound.

The rest of the paper is organized as follows. In Section 2, we review the definitions of weakly pseudoconvex locus and D'Angelo type, together with several examples. In Section

3, we prove the Whitney extension theorem needed for the construction. Local models with the prescribed weakly pseudoconvex loci are constructed in Section 4 in dimension two, and in Sections 5 and 6 in higher dimensions. To patch the local models with exterior barrier functions, we discuss a regularized maximum function in Section 7. In Sections 8–10, we construct global smoothly bounded domains and verify the desired properties, thereby completing the proofs of our main theorems. Finally, we close the paper in Section 11 by formulating two related conjectures concerning finite D'Angelo type, the degeneracy of the Levi form, and the size of the weakly pseudoconvex locus.

2 Preliminaries

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with a smooth defining function ρ, $n \geq 2$. Namely, ρ is a smooth real function near a neighborhood U of the boundary $b\Omega$ with $\Omega \cap U = \{p \in U : \rho(p) < 0\}$, $b\Omega \cap U = \{p \in U : \rho(p) = 0\}$ and $\nabla\rho \neq 0$ on $b\Omega$.

2.1 Levi determinant and weakly pseudoconvex locus

Definition 2.1. *At a boundary point $p \in b\Omega$, the complex normal gradient in terms of coordinates $(z_1, \dots, z_n)$ is given by the vector*

$$\partial\rho(p) = \left(\frac{\partial\rho}{\partial z_1}, \frac{\partial\rho}{\partial z_2}, \dots, \frac{\partial\rho}{\partial z_n}\right)\bigg|_p .$$

*The **complex tangent space** $T_p^{1,0}(b\Omega)$ is defined by*

$$T_p^{1,0}(b\Omega) = \left\{\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j} : \partial\rho(p)(\xi) = 0\right\} = \left\{\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j} : \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(p)\xi_j = 0\right\}.$$

After identifying $\xi = \sum_{j=1}^n \xi_j \frac{\partial}{\partial z_j}$ with its coefficient vector $(\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, the condition for the complex tangent space $T_p^{1,0}(b\Omega)$ can also be written as

$$\langle \bar\partial\rho(p), \xi\rangle_{\mathbb{C}^n} = 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ denotes the standard Hermitian inner product, taken to be linear in the second variable.

The intrinsic geometric behavior of the boundary is captured by the Levi form defined on the complex tangent space as below.

Definition 2.2. *Let $H_\rho = \left(\frac{\partial^2\rho}{\partial z_j \partial \bar z_k}\right)$ be the full $n \times n$ complex Hessian matrix of ρ. The **Levi form** at a point $p \in b\Omega$ evaluated on a vector $\xi \in T_p^{1,0}(b\Omega)$ is the quadratic form*

$$L_\rho(p; \xi, \bar\xi) = \xi^* H_\rho(p)\xi$$

restricted to the $(n-1)$-dimensional subspace $T_p^{1,0}(b\Omega)$.

Since the Levi form is intrinsically defined only on the complex tangent space, we shall use the complex Hessian restricted to this subspace. For $p \in b\Omega$, choose an $n \times (n-1)$ matrix E_p whose columns form an orthonormal basis of the complex tangent space $T_p^{1,0}(b\Omega)$ with respect to the standard Hermitian metric on $\mathbb{C}^n$. We define the **restricted complex Hessian** at p by

$$\tilde{H}_\rho(p) = E_p^* H_\rho(p) E_p.$$

This is an $(n-1) \times (n-1)$ Hermitian matrix representing the Levi form of $b\Omega$ at p with respect to the chosen orthonormal basis of $T_p^{1,0}(b\Omega)$.

Although the matrix $\widetilde{H}_\rho(p)$ depends on the choice of orthonormal basis, a different choice changes it only by unitary conjugation. Thus its eigenvalues are independent of this choice. We denote these restricted eigenvalues by

$$\lambda_1(p) \leq \lambda_2(p) \leq \cdots \leq \lambda_{n-1}(p).$$

The product

$$\prod_{j=1}^{n-1} \lambda_j(p) = \det \widetilde{H}_\rho(p)$$

will be called the **Levi determinant** of $b\Omega$ at p, with respect to the defining function ρ.

The following simple example illustrates how to compute the restricted complex Hessian and its eigenvalue(s).

Example 1. *Consider the pseudoconvex domain $\Omega = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 - 1 < 0\}$ with defining function $\rho(z) = z_1\bar{z}_1 + (z_2\bar{z}_2)^2 - 1$. We compute the restricted complex Hessian at the boundary point $p = (1, 0)$. The complex gradient is $\partial\rho = (\bar{z}_1, 2\bar{z}_2|z_2|^2)$, which at p becomes $(1, 0)$. It follows that the complex tangent space $T_p^{1,0}(b\Omega)$ is spanned by $E_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The full complex Hessian at p is*

$$H_\rho(p) = \begin{pmatrix} 1 & 0 \\ 0 & 4|z_2|^2 \end{pmatrix}\Bigg|_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the restricted Hessian matrix at p is

$$\tilde{H}_\rho(p) = E_p^* H_\rho(p) E_p = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

Hence the unique restricted eigenvalue is $\lambda_1 = 0$, identifying p as a point where the Levi determinant vanishes.

While the individual restricted eigenvalues $\lambda_j(p)$ are extrinsic and depend heavily on the scaling of the defining function, Sylvester's law of inertia guarantees that the signature of the Levi form is a biholomorphic invariant. In particular, the weak locus, the degeneracy of the Levi form as defined below, is independent of the choice of basis.

Definition 2.3. *The* ***weakly pseudoconvex locus*** *(or simply, the* ***weak locus****) of a smoothly bounded pseudoconvex domain Ω, denoted by $W \subset b\Omega$, is the set of all boundary points where the Levi determinant vanishes. Explicitly, it is the zero locus of the minimal restricted eigenvalue*

$$W := \{p \in b\Omega : \lambda_1(p) = 0\}.$$

2.2 Alternative definition by bordered Hessian

Computing restricted eigenvalues via point-wise orthonormal frames E_p is useful locally, but it is not well suited for global arguments. In general, the complex tangent bundle need not admit a global smooth orthonormal frame. We therefore replace the frame-dependent restricted Hessian by a globally defined bordered Hessian matrix.

Definition 2.4. *The **bordered Hessian matrix** H_ρ^{bd} associated with ρ is the $(n+1)\times(n+1)$ Hermitian matrix*

$$H_\rho^{bd} = \begin{pmatrix} 0 & \bar{\partial}\rho \\ (\partial\rho)^T & H_\rho \end{pmatrix} \quad \text{on} \quad b\Omega.$$

*We define the corresponding **bordered Levi determinant** by*

$$\Lambda_\rho = -\det H_\rho^{bd} \quad \text{on} \quad b\Omega.$$

The following known lemma shows that the bordered Levi determinant agrees with the Levi determinant up to multiplication by a smooth positive factor. For the reader's convenience, we provide the proof in Appendix A.

Lemma 2.5. *Let ρ be a smooth defining function for Ω. Then, for every $p \in b\Omega$, the bordered Levi determinant satisfies*

$$\Lambda_\rho(p) = |\partial\rho(p)|^2 \prod_{j=1}^{n-1} \lambda_j(p),$$

where $\lambda_1(p) \leq \cdots \leq \lambda_{n-1}(p)$ are the eigenvalues of the restricted complex Hessian $\widetilde{H}_\rho(p)$.

As an immediate consequence of Lemma 2.5, we have the following characterization of the weak locus in terms of the bordered Levi determinant.

Remark 2.6. *Since $|\partial\rho|$ is nowhere zero on the boundary of the smooth domain, the invariant Λ_ρ exhibits exactly the same vanishing property as the corresponding Levi determinant in view of Lemma 2.5. Consequently, the weakly pseudoconvex locus is characterized by*

$$W = \{p \in b\Omega : \Lambda_\rho(p) = 0\}.$$

We now examine the behavior of weak loci for two explicit examples below.

Example 2. *Consider the domain $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 - 1 < 0\}$. The defining function $\rho = |z|^2 + |w|^4 - 1$ is decoupled. By a direct computation, the bordered complex Hessian*

$$H_\rho^{bd}(z, w) = \begin{pmatrix} 0 & z & 2|w|^2 w \\ \overline{z} & 1 & 0 \\ 2|w|^2\overline{w} & 0 & 4|w|^2 \end{pmatrix}.$$

Hence by the cofactor expansion

$$\Lambda_\rho = -\det H_\rho^{bd} = z\left(4|w|^2\overline{z}\right) + 2|w|^2 w\left(2|w|^2\overline{w}\right) = 4|w|^2\left(|z|^2 + |w|^4\right).$$

Restricted on the boundary, we further have

$$\Lambda_\rho = 4|w|^2 \qquad \text{on} \quad b\Omega.$$

Therefore
$$\Lambda_\rho = 0 \quad \Longleftrightarrow \quad w = 0.$$
Combining this with the boundary equation gives $|z| = 1$*. Hence the weakly pseudoconvex locus is*
$$W = \{(z,0) : |z| = 1\} \cong_{\mathit{diff}} S^1.$$

In the second example below, the weakly pseudoconvex locus is diffeomorphic to $S^1 \times S^1$.

Example 3. *Consider the domain* $\Omega = \{(z,w) \in \mathbb{C}^2 : |w|^2 + 16\big(|z|^2 - 1\big)^4 - \frac{1}{4} < 0\}$*. For the defining function* $\rho = |w|^2 + 16\big(|z|^2-1\big)^4 - \frac{1}{4}$*, a straightforward computation gives the bordered complex Hessian*

$$H_\rho^{bd}(z,w) = \begin{pmatrix} 0 & 64(|z|^2-1)^3 z & w \\ 64(|z|^2-1)^3 \overline{z} & 64(|z|^2-1)^2(4|z|^2-1) & 0 \\ \overline{w} & 0 & 1 \end{pmatrix}.$$

Hence
$$\Lambda_\rho(z,w) = \ 64^2\,|z|^2(|z|^2-1)^6 + 64(|z|^2-1)^2(4|z|^2-1)|w|^2. \tag{2.1}$$

Since $16(|z|^2-1)^4 = \frac{1}{4} - |w|^2 \le \frac{1}{4}$ *on* $b\Omega$*, we have* $||z|^2 - 1| \le \frac{1}{2\sqrt{2}}$*, and in particular,* $4|z|^2 > 1$ *on* $b\Omega$*. Therefore both terms in (2.1) are nonnegative on* $b\Omega$*. Consequently, if* $\Lambda_\rho = 0$*, then the first term forces* $|z| = 1$*. The second term vanishes there as well. Substituting this into the boundary equation gives* $|w| = \frac{1}{2}$*. Thus the weakly pseudoconvex locus is*
$$W = \left\{(z,w) \in \mathbb{C}^2 : |z| = 1,\ |w| = \frac{1}{2}\right\} \cong_{\mathit{diff}} S^1 \times S^1.$$

2.3 Rigid domains

A domain $\Omega \subset \mathbb{C}^n$ is called rigid if there exists a real-valued function $f : \mathbb{C}^{n-1} \to \mathbb{R}$ such that
$$\Omega = \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \rho = \operatorname{Re} w + f(z,\bar{z}) < 0\}. \tag{2.2}$$
Since $\frac{\partial \rho}{\partial w} = \frac{1}{2}$ does not vanish, ρ is a defining function with nonvanishing gradient on $b\Omega$. Hence Ω has smooth boundary whenever f is smooth. Due to the independence along the $\mathrm{Im} w$ direction, such domains are never bounded.

We next compute the complex tangent space, the Levi form, and the weakly pseudoconvex locus of a rigid domain. They will be used repeatedly in our later construction.

Lemma 2.7. *Let* Ω *be a smooth rigid domain in* $\mathbb{C}^n$ *defined in (2.2). The following holds.*

1. *The complex tangent space at* $p \in b\Omega$ *is*

$$T_p^{1,0}(b\Omega) = \mathit{span}_{\mathbb{C}} \left\{ L_j = \frac{\partial}{\partial z_j} - 2 f_{z_j}(p) \frac{\partial}{\partial w} : j = 1, \ldots, n-1 \right\}.$$

2. *With respect to the complex tangential frame $\{L_1, \ldots, L_{n-1}\}$ as above, the restricted complex Hessian of ρ is exactly*

$$\tilde{H}_\rho = H_f = \left(f_{z_j \bar{z}_k}\right)_{j,k=1}^{n-1}.$$

In particular, Ω is pseudoconvex if and only if f is plurisubharmonic.

3. *If f is plurisubharmonic, then the weakly pseudoconvex locus is*

$$\begin{aligned} W &= \{p \in b\Omega : \det H_f(p) = 0\} \\ &= \{(z, -f(z,\bar{z}) + iv) \in \mathbb{C}^n : \det H_f(z) = 0, v \in \mathbb{R}\}. \end{aligned}$$

Proof. Since the complex normal gradient is

$$\partial\rho = \left(f_{z_1}, \ldots, f_{z_{n-1}}, \frac{1}{2}\right),$$

a $(1,0)$-vector $L = \sum_{j=1}^{n-1} \xi_j \frac{\partial}{\partial z_j} + \xi_n \frac{\partial}{\partial w}$ is complex tangential to $b\Omega$ if and only if

$$\sum_{j=1}^{n-1} f_{z_j}\xi_j + \frac{1}{2}\xi_n = 0.$$

In particular, for each $j = 1, \ldots, n-1$, setting $L_j = \frac{\partial}{\partial z_j} - 2f_{z_j}\frac{\partial}{\partial w}$, the above equation is satisfied. Since $\{L_j, j = 1, \ldots, n-1\}$ are linearly independent, they form a basis of the complex tangent space $T_p^{1,0}(b\Omega)$.

The complex Hessian of ρ has the block form

$$H_\rho = \begin{pmatrix} H_f & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_f = \left(f_{z_j\bar{z}_k}\right)_{j,k=1}^{n-1}.$$

Let E be the $n \times (n-1)$ matrix whose j-th column is the coefficient vector of L_j, $j = 1, \ldots, n-1$. Then

$$E = \begin{pmatrix} I_{n-1} \\ -2f_z \end{pmatrix}, \qquad f_z = \begin{pmatrix} f_{z_1} & \cdots & f_{z_{n-1}} \end{pmatrix}.$$

The restricted complex Hessian in the frame $\{L_1, \ldots, L_{n-1}\}$ is therefore

$$\tilde{H}_\rho = E^* H_\rho E = \begin{pmatrix} I_{n-1} & -2f_z^* \end{pmatrix} \begin{pmatrix} H_f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{n-1} \\ -2f_z \end{pmatrix} = H_f.$$

Thus $b\Omega$ is pseudoconvex precisely when f is plurisubharmonic. Moreover, the weakly pseudoconvex locus is given by

$$W = \{(z, w) \in b\Omega : \det H_f(z) = 0\},$$

provided that $H_f \geq 0$. $\square$

Alternatively, one can directly compute the corresponding bordered complex Hessian

$$\mathcal{H}_\rho^{\mathrm{bd}} = \begin{pmatrix} 0 & f_{\bar z} & \frac{1}{2} \\ f_z^T & H_f & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

The cofactor expansion gives the Levi determinant

$$\Lambda_\rho = -\det \mathcal{H}_\rho^{\mathrm{bd}} = \frac{1}{4}\det H_f.$$

from which the form of the weakly pseudoconvex locus follows.

2.4 D'Angelo type

The D'Angelo type for smooth hypersurfaces was introduced by D'Angelo to measure the maximum order of contact of complex analytic varieties with the hypersurfaces; see, for example, [2, 3]. We first recall the notion of vanishing order. Let $g = (g_1, \ldots, g_N)$ be a smooth vector-valued function defined near $0 \in \mathbb{C}$, with $g(0) = 0$. We define

$$\nu(g) = \min_{1 \le j \le N} \nu(g_j).$$

Equivalently, $\nu(g) = m$ if

$$g(\zeta) = P_m(\zeta, \bar\zeta) + O(|\zeta|^{m+1}),$$

where P_m is a nonzero homogeneous polynomial vector of degree m in ζ and $\bar\zeta$.

Definition 2.8. *Let M be a smooth hypersurface in $\mathbb{C}^n$ with smooth defining function ρ. The **D'Angelo type** of M at a point $p \in M$ is defined by*

$$\tau(p) := \sup_\gamma \frac{\nu(\rho \circ \gamma)}{\nu(\gamma - p)}, \tag{2.3}$$

*where the supremum is taken over all germs of nonconstant holomorphic curves $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$. The point p is called a point of **finite type** if $\tau(p) < \infty$. Otherwise, it is called a point of infinite type.*

The hypersurface M is said to be of finite D'Angelo type if

$$\sup_{p \in M} \tau(p) < \infty.$$

A smoothly bounded domain Ω in $\mathbb{C}^n$ is said to be of finite D'Angelo type if its boundary $b\Omega$ is of finite D'Angelo type.

For a smoothly bounded pseudoconvex domain Ω and a boundary point p, the condition $\tau(p) = 2$ is equivalent to strict pseudoconvexity at p, where the Levi form is strictly positive. Hence the weakly pseudoconvex locus W can be equivalently described in terms of the D'Angelo type as

$$W = \{p \in b\Omega : \tau(p) > 2\}. \tag{2.4}$$

If the boundary $b\Omega$ contains a germ of a nonconstant holomorphic curve, then the corresponding D'Angelo type is infinite everywhere on the curve. As a consequence of (2.4), the image of the whole curve is also contained in the weakly pseudoconvex locus W, where the Levi determinant vanishes. In the real-analytic category, this connection is especially rigid. D'Angelo proved in [2] that, if $b\Omega$ is real-analytic near p, then $\tau(p) = \infty$ if and only if $b\Omega$ contains a germ of a nonconstant holomorphic curve passing through p. This equivalence fails in general for smooth hypersurfaces, as shown by a simple example below.

Example 4. *Let $M := \{(z, w) \in \mathbb{C}^2 : \rho = \operatorname{Re} w + e^{-\frac{1}{|z|^2}} = 0\}$, where $e^{-\frac{1}{|z|^2}}$ is understood to be extended smoothly by 0 at $z = 0$. Since $\rho \circ \gamma_0 = e^{-\frac{1}{|\zeta|^2}}$ along the complex line $\gamma_0(\zeta) = (\zeta, 0)$ and $e^{-\frac{1}{|\zeta|^2}}$ vanishes to infinite order at 0, it follows that 0 is a point of infinite D'Angelo type.*

On the other hand, M contains no nonconstant germ of a holomorphic curve through 0. Indeed, if $\gamma(\zeta) = (z(\zeta), w(\zeta))$ is such a curve, then $z(0) = w(0) = 0$ and

$$\operatorname{Re} w(\zeta) = -e^{-\frac{1}{|z(\zeta)|^2}} \quad \text{for all} \quad \zeta \text{ near } 0.$$

For every positive integer k, taking k derivatives with respect to ζ on both sides and then setting $\zeta = 0$, one immediately obtains that the holomorphic function

$$w(\zeta) \equiv 0.$$

Consequently, $e^{-\frac{1}{|z(\zeta)|^2}} \equiv 0$, and hence

$$z(\zeta) \equiv 0.$$

Therefore γ is constant, proving that no nonconstant holomorphic curve germ is contained in M.

Thus, a point of infinite D'Angelo type on a hypersurface does not in general imply the existence of a complex analytic subvariety contained in the hypersurface. Moreover, Kim–Thu [12, Example 2] and Nguyen–Chu [15] constructed some smooth real hypersurfaces with points of infinite type at which the supremum in (2.3) of Definition 2.8 is not attained. Equivalently, such hypersurfaces admit no holomorphic curve having infinite order contact with the hypersurface at a point of infinite type. We also refer to [7] for pseudoconvex examples of this phenomenon. On the other hand, Fornæss, Lee and the last author [6] proved the existence of formal complex analytic curves at points of infinite type.

Despite the simplicity of the definition of D'Angelo type, determining whether a hypersurface is of finite type can be technically difficult in practice, even in the real-analytic setting in $\mathbb{C}^2$, due to the involvement of the implicit function theorem. See Appendix B for an elementary-looking non-algebraic real-analytic hypersurface in $\mathbb{C}^2$ for which determining finite D'Angelo type appears to be nontrivial.

3 Whitney extension theorem

The classical Whitney extension theorem may be stated as follows. Let E be a closed subset of $\mathbb{R}^n$. Suppose that $\{f_\alpha\}$ is a collection of functions on E, indexed by all multi-indices α, satisfying the following compatibility condition: for every $m \in \mathbb{N}$ and every multi-index α with $|\alpha| \leq m$,

$$f_\alpha(x) = \sum_{|\beta| \leq m-|\alpha|} \frac{f_{\alpha+\beta}(y)}{\beta!}(x-y)^\beta + R_\alpha(x,y) \qquad \text{for all} \;\; x, y \in E,$$

where the function $R_\alpha(x,y) = o\left(|x-y|^{m-|\alpha|}\right)$ uniformly as $x \to y$ in E. Then there exists a smooth function F on $\mathbb{R}^n$ such that

$$F = f_0 \quad \text{on } E, \qquad D^\alpha F = f_\alpha \quad \text{on } E,$$

for every multi-index α. See, for instance, [8, 13, 14, 17].

We shall need the following variant of the Whitney extension theorem for existence of nonnegative smooth functions that are flat on a given closed set E in the later construction. Recall that a smooth function f is said to be flat at a point if all its derivatives vanish at that point.

Proposition 3.1. *Let E be a closed subset of $\mathbb{R}^n$. Then there exists a smooth function ψ on $\mathbb{R}^n$ such that*

1. *$\psi \geq 0$ on $\mathbb{R}^n$;*
2. *$\psi^{-1}(0) = E$;*
3. *ψ is flat at every point of E.*

Proof. Since $\mathbb{R}^n$ is paracompact (i.e., every open cover has a locally finite refinement) and $\mathbb{R}^n \setminus E$ is open, there exists a countable locally finite collection of open balls $\{B_{r_j}(c_j)\}_{j\in\mathbf{N}}$ in $\mathbb{R}^n$ centered at c_j with radius $r_j > 0$, such that

$$\mathbb{R}^n \setminus E = \bigcup_{j=1}^{\infty} B_{r_j}(c_j).$$

See Lemma A.1. Here, local finiteness means that every point has a neighborhood intersecting only finitely many balls in the cover.

For each $j \in \mathbb{N}$, define

$$\psi_j(x) = \begin{cases} e^{\frac{1}{|x-c_j|^2 - r_j^2}}, & x \in B_{r_j}(c_j); \\ 0, & \text{otherwise.} \end{cases}$$

Then $\psi_j \in C^\infty(\mathbb{R}^n)$, $\psi_j > 0$ on $B_{r_j}(c_j)$ and is flat everywhere else. Choose constants $a_j > 0$ such that

$$a_j \sup_{|x|<j} \sum_{\alpha\in\mathbb{N}^n, |\alpha|\leq j} |D^\alpha \psi_j(x)| \leq 2^{-j}, \tag{3.1}$$

and set

$$\psi := \sum_{j=1}^{\infty} a_j \psi_j.$$

By the local finiteness of the covering, the function ψ is smooth on $C^\infty(\mathbb{R}^n \setminus E)$.

To see that ψ is smooth across E, we show that for each $R > 0$ and multi-index $\lambda \in \mathbb{N}^n$,

$$\sum_{j=1}^{\infty} \sup_{|x|<R} |a_j D^\lambda \psi_j(x)|$$

converges uniformly on $\{|x| < R\}$. Indeed, let $N \in \mathbb{N}$ be such that $N \geq \max\{R, |\lambda|\}$. For $q > m > N$ we have

$$\begin{aligned}\sum_{j=m+1}^{q} \sup_{|x|<R} |a_j D^\lambda \psi_j(x)| &\leq \sum_{j=m+1}^{q} a_j \sup_{|x|<N} \sum_{\nu\in\mathbb{N}^n, |\nu|\leq N} |D^\nu \psi_j(x)| \\ &\leq \sum_{j=m+1}^{q} a_j \sup_{|x|<j} \sum_{\nu\in\mathbb{N}^n, |\nu|\leq j} |D^\nu \psi_j(x)| \leq \sum_{j=m+1}^{q} 2^{-j},\end{aligned}$$

where the last step follows from (3.1). Hence the derivative series converges uniformly on compact subsets of $\mathbb{R}^n$. Therefore $\psi \in C^\infty(\mathbb{R}^n)$ and

$$D^\lambda \psi = \sum_{j=1}^{\infty} a_j D^\lambda \psi_j \tag{3.2}$$

for every multi-index λ.

We now verify the desired properties of ψ. Since each $\psi_j \geq 0$ and each $a_j > 0$, we have $\psi \geq 0$ on $\mathbb{R}^n$. Moreover,

$$\psi(p) = 0 \iff \psi_j(p) = 0 \text{ for all } j \iff p \notin B_{r_j}(c_j) \text{ for all } j \iff p \in E.$$

Thus $\psi^{-1}(0) = E$. Finally, if $p \in E$, then $p \notin B_{r_j}(c_j)$ for every j, and hence each ψ_j is flat at p. The flatness of ψ on $p \in E$ then follows from this together with (3.2). □

From the proof, one also sees that the function ψ constructed above is real-analytic at every point of

$$\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} bB_{r_j}(c_j),$$

and is not real-analytic elsewhere.

4 Unbounded convex models in dimension two

In this section, we shall construct a convex domain in $\mathbb{C}^2$ whose weak locus forms a trivial $\mathbb{R}$-family over the prescribed closed set $E \subset \mathbb{R}$, and hence is diffeomorphic to $E \times \mathbb{R}$. Note that E may not be real-analytic. Thus we ought to carry out the construction entirely within the smooth category.

Recall that a domain $\Omega \subset \mathbb{R}^n$ is called **convex** if, for any two points in Ω, the line segment joining them is contained in Ω. For a domain $\Omega \subset \mathbb{C}^n$, convexity is understood after identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. A C^2 function ρ on a convex domain $D \subset \mathbb{R}^n$ is called **convex** if its real Hessian $D^2\rho$ is positive semi-definite at every point of D. In particular, a domain defined as a sublevel set of a convex function on a convex ambient domain is convex.

Proposition 4.1. *Let $E \subset \mathbb{R} \subset \mathbb{C}$ be a closed set. For any integer $m \geq 2$, there exists a smooth convex domain $\Omega \subset \mathbb{C}^2$ such that the following hold.*

1. *The weakly pseudoconvex locus W of Ω is exactly*

$$W = \{(z, w) \in b\Omega : \ \operatorname{Re} z \in E, \ \operatorname{Im} z = 0\} \cong_{\mathit{diff}} E \times \mathbb{R};$$

2. *The D'Angelo type satisfies*

$$\tau(p) = 2m \qquad \textit{for every} \quad p \in W.$$

Proof. We shall define a rigid domain Ω by

$$\Omega = \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + f(z, \bar{z}) < 0\}.$$

Here in terms of coordinates $z = x + iy$ and $w = u + iv$, the potential function f will be chosen in the decoupled form

$$f(z, \bar{z}) = y^{2m} + \Phi(x), \tag{4.1}$$

where $m \in \mathbb{Z}^+$, and $\Phi : \mathbb{R} \to \mathbb{R}$ is a smooth function. Note that such a domain is always smooth but unbounded.

Choice of Φ. For the given closed set $E \subset \mathbb{R}$, we shall construct Φ such that $\Phi''(x) = 0$ exactly on E and $\Phi''(x) > 0$ everywhere else. In fact, by the Whitney extension theorem Proposition 3.1, there exists a smooth, non-negative function ψ on $\mathbb{R}$ such that

$$\psi(x) = 0 \iff x \in E, \tag{4.2}$$

and ψ is flat at each point of E. Integrate ψ twice and set

$$\Phi(x) = \int_0^x \int_0^t \psi(s)\, ds\, dt. \tag{4.3}$$

By the Fundamental Theorem of Calculus,

$$\Phi'' = \psi \quad \text{on} \quad \mathbb{R}.$$

Thus Φ satisfies the desired properties. Moreover, since

$$\Phi' = \int_0^x \psi(t)\, dt$$

and $\psi(x) \geq 0$, inspecting separately the cases $x < 0$ and $x \geq 0$ gives

$$x\Phi'(x) \geq 0 \quad \text{on} \quad \mathbb{R}. \tag{4.4}$$

This property will be crucially used in Sections 8 and 9.

Convexity. In the real coordinate frame $(x,y,u,v)\in\mathbb{R}^4$, the real Hessian matrix of the defining function $\rho(z,w):=\operatorname{Re}w+f(z,\bar z)$ is

$$D^2\rho(x,y,u,v)=\begin{pmatrix}\psi(x) & 0 & 0 & 0\\ 0 & 2m(2m-1)y^{2m-2} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

which is positive semi-definite for all $m\geq 2$. Thus, Ω is a sublevel set of the convex function ρ in $\mathbb{R}^4$, and hence is a convex domain. In particular, Ω is pseudoconvex.

Weak locus. According to Lemma 2.7, the weak locus of Ω is

$$W=\{(z,w)\in b\Omega : f_{z\bar z}(z)=0\}.$$

By a direct computation, one has

$$f_{z\bar z}=\frac{1}{4}\Delta f=\frac{1}{4}\left[2m(2m-1)y^{2m-2}+\psi(x)\right].$$

Since both terms in the bracket are nonnegative, the Levi form vanishes if and only if

$$y=0\quad\text{and}\quad\psi(x)=0.$$

Combining this with (4.2), one has the weak locus

$$\begin{aligned}W=&\{(z,w)\in b\Omega : \operatorname{Re}z\in E, \operatorname{Im}z=0\}\\ =&\{(x,0,-\Phi(x),v)\in\mathbb{R}^4 : x\in E\}\cong_{\text{diff}} E\times\mathbb{R}.\end{aligned}$$

D'Angelo type. At points off the weak locus W, since Ω is strictly pseudoconvex, the D'Angelo type is equal to 2. To calculate the type effectively at a fixed point

$$p=(z_p,w_p)=(x_p+i0,-\Phi(x_p)+iv_p)\in W$$

where $x_p\in E$, we need to normalize the coordinates system.

Since $\Phi''(x)=\psi(x)$ and ψ is flat at $x_p\in E$, the formal Taylor series of Φ around the center x_p terminates exactly after the linear term, leaving a smooth remainder function $R(x)$ that is flat at x_p:

$$\Phi(x)=\Phi(x_p)+\Phi'(x_p)(x-x_p)+R(x).$$

Substituting this expansion back into our defining function yields

$$\rho(z,w)=\operatorname{Re}\,w+y^{2m}+\Phi(x_p)+\Phi'(x_p)(x-x_p)+R(x).$$

Noting that x_p, $\Phi(x_p)$ and $\Phi'(x_p)$ are real, we collect the linear and constant terms into a single pluriharmonic expression:

$$\operatorname{Re}\,w+\Phi(x_p)+\Phi'(x_p)\operatorname{Re}(z-z_p)=\operatorname{Re}\left[w+\Phi(x_p)+\Phi'(x_p)(z-z_p)\right].$$

We define a global biholomorphic coordinate transformation $(z, w) \to (\tilde{z}, \tilde{w})$ via

$$\begin{aligned} \tilde{z} &= z - z_p; \\ \tilde{w} &= w + \Phi(x_p) + \Phi'(x_p)(z - z_p). \end{aligned}$$

In these normalized coordinates, the defining function simplifies to

$$\tilde{\rho}(\tilde{z}, \tilde{w}) = \mathrm{Re}\ \tilde{w} + \tilde{y}^{2m} + \tilde{R}(\tilde{x}),$$

where $\tilde{R}(\tilde{x})$ is flat at $\tilde{x} = 0$.

Let

$$\gamma(\zeta) = (\tilde{z}(\zeta), \tilde{w}(\zeta))$$

be a nonconstant holomorphic curve with $\gamma(0) = 0$. Restricting $\tilde{\rho}$ on γ, we have

$$(\tilde{\rho} \circ \gamma)(\zeta) = \mathrm{Re}\, \tilde{w}(\zeta) + |\, \mathrm{Im}\, \tilde{z}(\zeta)|^{2m} + \tilde{R}(\mathrm{Re}\, \tilde{z}(\zeta)).$$

Since $\tilde{R}$ is flat at 0, the pullback $\tilde{R}(\mathrm{Re}\, \tilde{z}(\zeta))$ vanishes to infinite order. Thus it does not affect the D'Angelo type.

If $\nu(\tilde{w}(\zeta)) < \nu(\tilde{z}(\zeta))$, then $\mathrm{Re}\, \tilde{w}(\zeta)$ gives the leading term of $\tilde{\rho} \circ \gamma$, and hence

$$\frac{\nu(\tilde{\rho} \circ \gamma)}{\nu(\gamma)} = 1.$$

We next consider $\nu(\tilde{w}(\zeta)) \geq \nu(\tilde{z}(\zeta))$. The leading nonharmonic term in $(\mathrm{Im}\, \tilde{z}(\zeta))^{2m}$ has order $2m\nu(\tilde{z}(\zeta))$. The term $\mathrm{Re}\, \tilde{w}(\zeta)$ is harmonic, and therefore can not cancel the leading nonharmonic term in $(\mathrm{Im}\, \tilde{z}(\zeta))^{2m}$. Hence $\nu(\tilde{\rho} \circ \gamma) \leq 2m\nu(\tilde{z}(\zeta))$, and

$$\frac{\nu(\tilde{\rho} \circ \gamma)}{\nu(\gamma)} \leq \frac{2m\nu(\tilde{z}(\zeta))}{\nu(\tilde{z}(\zeta))} = 2m.$$

On the other hand, taking $\gamma(\zeta) = (\zeta, 0)$, we obtain

$$(\tilde{\rho} \circ \gamma)(\zeta) = (\mathrm{Im}\, \zeta)^{2m} + \tilde{R}(\mathrm{Re}\, \zeta).$$

Altogether, we have shown that the D'Angelo type at 0 is

$$\tau(0) = 2m.$$

The proof is complete. □

There are two remarks in order concerning the proof of Proposition 4.1.

1. As shown in the proof of Proposition 4.1, the flatness of ψ at points in E makes it necessary to add the extra convex term $|y|^{2m}$ to the potential f in (4.1). This term ensures that the boundary has finite D'Angelo type. Without it, the complex tangent line would have infinite type at every point lying over E.

2. Alternatively, since the domain Ω is convex, the D'Angelo type can be checked by the corresponding order of contact by complex lines according to a theorem of Boas-Straube [1]. We leave the details to the interested reader.

5 Local pseudoconvex models in higher dimensions

In this section, we construct local pseudoconvex domains in $\mathbb{C}^n$ with prescribed weak loci. The construction relies on a local existence theorem of Kallel-Jallouli [10] for smooth plurisubharmonic solutions to the complex Monge–Ampère equation, which is used to obtain the required potential function. The verification of finite D'Angelo type is technically involved and is ultimately reduced to controlling the vanishing order of the nonharmonic terms of the potential function along holomorphic curves. A key ingredient in this control is Hadamard's inequality, which bounds the determinant of a positive semidefinite Hermitian matrix by the product of its diagonal entries.

Theorem 5.1. *[10] Let g be a smooth nonnegative function defined in a neighborhood of $z_0 \in \mathbb{C}^d, d \geq 1$. Suppose that g is not flat at z_0. Then there exist a neighborhood U of z_0 and a real-valued smooth plurisubharmonic function f on U such that*

$$\det H_f(z, \bar{z}) = g(z, \bar{z}) \quad \text{on} \quad U.$$

Proposition 5.2. *Let $E \subset \mathbb{R}^{n-1} \subset \mathbb{C}^{n-1}$ be a closed set. For any integer $m \geq 2$ and every point $x_0 \in E$, there exist a neighborhood U of $x_0 + i0$ in $\mathbb{C}^{n-1}$ and a hypersurface M defined in $U \times \mathbb{C} \subset \mathbb{C}^n$ such that the following holds.*

1. *M bounds a pseudoconvex side.*

2. *The weakly pseudoconvex locus W of M is precisely*

$$W = \{(z, w) \in M : \operatorname{Re} z \in E, \ \operatorname{Im} z = 0\}.$$

3. *M is of finite D'Angelo type, with type less than or equal to $2m + 2$.*

Proof. Since E is a closed set in $\mathbb{R}^{n-1}$, the Whitney extension theorem Proposition 3.1 yields a function $\psi \in C^\infty(\mathbb{R}^{n-1})$ such that

$$\psi \geq 0, \qquad \psi^{-1}(0) = E,$$

and ψ is flat at every point of E. Write $z = x + iy$, with $x, y \in \mathbb{R}^{n-1}$, and consider the nonnegative smooth function

$$|y|^{2m} + \psi(x).$$

Given $x_0 \in E$, since y^{2m} has finite order vanishing at $y = 0$, the function $|y|^{2m} + \psi(x)$ is not flat at $z_0 = x_0 + i0 \in \mathbb{C}^{n-1}$. Therefore, one can apply Theorem 5.1 by Kallel-Jallouli to obtain a smooth plurisubharmonic function f on a neighborhood U of z_0 in $\mathbb{C}^{n-1}$ such that

$$\det H_f(z, \bar{z}) = |y|^{2m} + \psi(x) \quad \text{on} \quad U. \tag{5.1}$$

In particular, the complex Hessian $H_f \geq 0$. The additional term $|y|^{2m}$ is essential here: it not only allows one to apply the existence theorem of Kallel-Jallouli, but also provides the finite type control needed along the weakly pseudoconvex locus, as will be seen later.

Define the hypersurface by

$$M = \{(z, w) \in U \times \mathbb{C} : \rho = \operatorname{Re} w + f(z, \bar{z}) = 0\}.$$

Since the defining function is plurisubharmonic, M bounds a pseudoconvex side.

Weak locus. Both terms on the right-hand side of (5.1) are nonnegative. Thus for all $z \in U$,

$$\det H_f(z, \bar{z}) = 0 \qquad \Longleftrightarrow \qquad y = 0,\ x \in E.$$

Hence by Lemma 2.7, the stated description of W holds.

D'Angelo type. Fix a point $p = (z_p, w_p) \in W$, where $z_p = x_p + i0$ with $x_p \in E$, and $w_p = -f(z_p, \bar{z}_p) + iv_p$. Replacing w by $w + w_p$ which also subtracts the constant $f(z_p, \bar{z}_p)$ from f, we may assume that

$$p = (z_p, 0) = (x_p + i0, 0) \qquad \text{and} \qquad f(z_p, \bar{z}_p) = 0.$$

Let $\gamma(\zeta) = (z(\zeta), w(\zeta))$ be a nonconstant holomorphic curve with $\gamma(0) = p$. Set

$$\delta = \nu(z(\zeta) - z_p), \qquad \alpha = \nu(w(\zeta)).$$

Then

$$\nu(\gamma) = \min\{\delta, \alpha\}. \tag{5.2}$$

The pullback of ρ by γ is

$$\rho(\gamma(\zeta)) = \operatorname{Re} w(\zeta) + f(z(\zeta), \overline{z(\zeta)}).$$

If the order $\delta = \infty$, then $z \equiv z_p$ and $\nu(\rho \circ \gamma) = \nu(\operatorname{Re} w(\zeta)) = \alpha$. Hence

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} = 1.$$

In what follows, we assume that $\delta < \infty$.

Decompose $f(z(\zeta), \overline{z(\zeta)})$ into a sum of a harmonic term, and a non-harmonic term. Denote by L the vanishing order of the nonharmonic term of $f(z(\zeta), \overline{z(\zeta)})$. Indeed, for $N \geq 1$, write the Taylor polynomial of $f(z(\zeta), \overline{z(\zeta)})$ at 0, of total degree at most N, as $P_N(\zeta, \overline{\zeta}) = \sum_{p+q \leq N} a_{pq} \zeta^p \overline{\zeta}^q$. Let Q_N denote the mixed nonharmonic part of P_N:

$$Q_N(\zeta, \overline{\zeta}) = \sum_{\substack{p,q \geq 1 \\ p+q \leq N}} a_{pq} \zeta^p \overline{\zeta}^q.$$

Define

$$L_N = \begin{cases} \min\{p + q : p, q \geq 1,\ a_{pq} \neq 0\}, & Q_N \not\equiv 0; \\ N + 1, & Q_N \equiv 0. \end{cases}$$

Then

$$L = \lim_{N \to \infty} L_N \in \mathbb{N} \cup \{\infty\}.$$

We claim that L is finite. Indeed, since $x_p \in E$ and ψ is flat at x_p, the term $\psi(x(\zeta))$ vanishes to infinite order at 0. By the definition of δ, at least one component of $z(\zeta) - z_p$ has order δ. Since $z(\zeta)$ is holomorphic, the function $|y(\zeta)|^2$ has order 2δ. Therefore

$$\nu(\det H_f(z(\zeta), \overline{z(\zeta)})) = \nu(|y(\zeta)|^{2m} + \psi(x(\zeta))) = \nu(|y(\zeta)|^{2m}) = 2m\delta. \tag{5.3}$$

Suppose, toward a contradiction, that $L = \infty$, meaning that all non-harmonic terms of $F(\zeta) := f(z(\zeta), \overline{z(\zeta)})$ vanish to infinite order. In particular, $F_{\zeta\bar{\zeta}}$ is flat at 0. By the chain rule,

$$F_{\zeta\bar{\zeta}}(\zeta) = \sum_{j,k=1}^{n-1} f_{z_j\bar{z}_k}(z(\zeta))z_j'(\zeta)\overline{z_k'(\zeta)} = \left\langle H_f(z(\zeta), \overline{z(\zeta)})z'(\zeta), z'(\zeta) \right\rangle.$$

Let $\lambda_{\min}(\zeta)$ be the smallest eigenvalue of $H_f(z(\zeta), \overline{z(\zeta)})$. Since $H_f(z(\zeta), \overline{z(\zeta)}) \geq 0$, for $\zeta \neq 0$ we have

$$0 \leq \lambda_{\min}(\zeta) \leq \frac{\left\langle H_f(z(\zeta), \overline{z(\zeta)})z'(\zeta), z'(\zeta) \right\rangle}{\|z'(\zeta)\|^2} = \frac{F_{\zeta\bar{\zeta}}(\zeta)}{\|z'(\zeta)\|^2}.$$

The numerator is flat, while $\|z'(\zeta)\|^2$ is of vanishing order $2(\delta - 1)$. Thus

$$\lambda_{\min}(\zeta) = O(|\zeta|^k) \qquad \text{for all} \quad k \geq 0.$$

On the other hand, the remaining eigenvalues of $H_f(z(\zeta), \overline{z(\zeta)})$ are locally bounded. Indeed, since the matrix H_f is positive semi-definite by pseudoconvexity, its eigenvalues are nonnegative and bounded above by the trace, which is continuous and therefore locally bounded. Consequently, $\det H_f(z(\zeta), \overline{z(\zeta)})$ is flat at 0. This is a contradiction to (5.3). Hence the claim is proved.

Fix an integer $N > L + 2$. Decompose the Taylor expansion of $f(z(\zeta), \overline{z(\zeta)})$ up to order N as

$$f(z(\zeta), \overline{z(\zeta)}) = h(\zeta) + g(\zeta) + O(|\zeta|^{N+1}),$$

where the polynomials $h(\zeta)$ and $g(\zeta)$ denote the corresponding harmonic part and the non-harmonic part, respectively. In particular, since $f(z_p, \bar{z}_p) = 0$,

$$\nu(g(\zeta)) \geq \nu(z(\zeta) - z_p) = \delta \qquad \text{and} \qquad \nu(h(\zeta)) \geq \nu(z(\zeta) - z_p) = \delta. \tag{5.4}$$

Since $h(\zeta)$ is harmonic, there exists a holomorphic polynomial $H(\zeta)$ with $H(0) = 0$ such that

$$h(\zeta) = \operatorname{Re} H(\zeta),$$

and thus

$$\rho(\gamma(\zeta)) = \operatorname{Re}(w(\zeta) + H(\zeta)) + g(\zeta) + O(|\zeta|^{N+1}).$$

Note that by definition of L and (5.4),

$$L = \nu(g(\zeta)) \geq \delta \qquad \text{and} \qquad \nu(H(\zeta)) = \nu(h(\zeta)) \geq \delta. \tag{5.5}$$

Since a pure term cannot cancel a mixed term, we obtain

$$\nu(\rho \circ \gamma) = \min\{\beta, L\},$$

where

$$\beta = \nu\big(\operatorname{Re}(w(\zeta) + H(\zeta))\big) = \nu\big(w(\zeta) + H(\zeta)\big).$$

Consequently, together with (5.2) one has

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} = \frac{\min\{\beta, L\}}{\min\{\alpha, \delta\}}. \tag{5.6}$$

We now estimate (5.6) by splitting into two cases. First suppose that $\delta > \alpha$. Then

$$\nu(\gamma) = \min\{\alpha, \delta\} = \alpha.$$

Since $w(\zeta)$ has order α, whereas $H(\zeta)$ has order at least δ by (5.5), no cancellation at order α is possible for $w(\zeta) + H(\zeta)$. Hence

$$\beta = \alpha.$$

Consequently,

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} = \frac{\min\{\alpha, L\}}{\alpha} \leq 1.$$

Now suppose that $\alpha \geq \delta$. Then (5.6) is reduced to

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} = \frac{\min\{\beta, L\}}{\delta} \leq \frac{L}{\delta}. \tag{5.7}$$

To estimate the vanishing order L, the nonharmonic part of $f(z(\zeta), \overline{z(\zeta)})$, we compare it with that of the diagonal entries $f_{z_j\bar{z}_j}(z(\zeta), \overline{z(\zeta)})$ of $H_f(z(\zeta), \overline{z(\zeta)})$. We first claim that for every $j = 1, \ldots, n-1$,

$$\nu(f_{z_j\bar{z}_j}(z(\zeta), \overline{z(\zeta)})) \geq L - 2\delta. \tag{5.8}$$

Indeed, write a typical mixed monomial in the Taylor expansion of f at z_p in the form

$$(z - z_p)^\alpha (\bar{z} - \bar{z}_p)^\beta, \qquad |\alpha| \geq 1, \quad |\beta| \geq 1.$$

Such monomials are precisely the terms which contribute to the non-harmonic part. If this monomial contains both $z_j - z_{0,j}$ and $\bar{z}_j - \bar{z}_{0,j}$, so that $\alpha_j \geq 1$ and $\beta_j \geq 1$, applying $\partial_{z_j}\partial_{\bar{z}_j}$ lowers the two exponents α_j and β_j by one. Hence the total bidegree is lowered by two. After restricting to the curve $z(\zeta)$, the order in ζ is therefore lowered by at most 2δ. On the other hand, if the monomial does not contain both $z_j - z_{p,j}$ and $\bar{z}_j - \bar{z}_{p,j}$, then $\partial_{z_j}\partial_{\bar{z}_j}$ annihilates that monomial. The first nonzero term which survives after applying $\partial_{z_j}\partial_{\bar{z}_j}$ must therefore come from either another lowest-order mixed monomial or from a mixed monomial of higher order. In either case its order along the curve is no smaller than $L - 2\delta$. The claim is confirmed.

On the other hand, for a positive semi-definite Hermitian matrix, Hadamard's inequality bounds the determinant from above by the product of its diagonal entries. We apply this to the matrix $H_f(z(\zeta), \overline{z(\zeta)})$ to get

$$0 \leq \det H_f(z(\zeta), \overline{z(\zeta)}) \leq \prod_{j=1}^{n-1} f_{z_j\bar{z}_j}(z(\zeta), \overline{z(\zeta)}), \qquad |\zeta| << 1.$$

Taking vanishing orders of the above and making use of (5.3) further give

$$\nu\left(\prod_{j=1}^{n-1} f_{z_j\bar{z}_j}(z(\zeta), \overline{z(\zeta)})\right) \leq 2m\delta.$$

Then there exists $k \in \{1, \ldots, n-1\}$ such that

$$\nu(f_{z_k\bar{z}_k}(z(\zeta), \overline{z(\zeta)})) \leq 2m\delta.$$

Combining (5.8) with the index k chosen above gives

$$L - 2\delta \le \nu(f_{z_k\bar{z}_k}(z(\zeta), \overline{z(\zeta)})) \le 2m\delta.$$

Thus

$$L \le (2m+2)\delta.$$

Finally, invoking (5.7) one infers

$$\frac{\nu(\rho \circ \gamma)}{\nu(\gamma)} \le 2m+2.$$

Taking the supremum over all such curves gives $\tau(p) \le 2m+2$. □

6 Local convex models in higher dimensions

To construct convex domains with desired weak loci, we have to compromise by lowering one real dimension on the prescribed closed set. Our tool for constructing the potential is the following special case of Hong–Zuily's result for the real degenerate Monge–Ampère equation.

Theorem 6.1. *[9] Let ψ be a smooth nonnegative function defined in a neighborhood of $x_0 \in \mathbb{R}^d, d \ge 1$. Suppose that ψ is not flat at x_0. Then there exist a convex neighborhood U of x_0 and a smooth convex function to*

$$\det D^2\Phi = \psi \quad on \quad U.$$

Proposition 6.2. *Let $E \subset \mathbb{R}^{n-2} \subset \mathbb{C}^{n-1}$ be a closed set. For every integer $m \ge 2$ and every point $\hat{x}_0 \in E$, there exist a neighborhood U of $(\hat{x}_0, 0)$ in $\mathbb{R}^{n-1}$ and a hypersurface M defined in $(U + i\mathbb{R}^{n-1}) \times \mathbb{C} \subset \mathbb{C}^n$, such that the following holds.*

1. *M bounds a convex side.*
2. *The weakly pseudoconvex locus W of M is precisely*

$$W = \{(z, w) \in M : \operatorname{Re} z \in E \times \{0\},\ \operatorname{Im} z = 0\}.$$

3. *M is of finite D'Angelo type, with type less than or equal to $2m$.*

Proof. As in the previous sections, we apply the Whitney extension theorem Proposition 3.1 to obtain a smooth nonnegative function ψ characterizing the set E, such that (4.2) holds and ψ is flat at each point of E. For each point $\hat{x}_0 \in E \subset \mathbb{R}^{n-2}$, we invoke the local existence theorem of Hong–Zuily, Theorem 6.1, to obtain a smooth convex potential Φ and a convex neighborhood of U of $(\hat{x}_0, 0)$ in $\mathbb{R}^{n-1}$ such that

$$\det D^2\Phi(x) = |x_{n-1}|^{2m} + \psi(\hat{x}) \quad \text{on} \quad U, \tag{6.1}$$

where $x = (\hat{x}, x_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}$. Set

$$f(z, \bar{z}) := |y|^{2m} + \Phi(x),$$

where $y \in \mathbb{R}^{n-1}$ and $z = x+iy \in \mathbb{C}^{n-1}$. Then f is convex on $U+i\mathbb{R}^{n-1} \subset \mathbb{C}^{n-1}$. Consequently, the rigid hypersurface

$$M = \{(z, w) \in (U + i\mathbb{R}^{n-1}) \times \mathbb{C} : \operatorname{Re} w + f(z, \bar{z}) = 0\}$$

bounds the convex side

$$\Omega = \{(z, w) \in (U + i\mathbb{R}^{n-1}) \times \mathbb{C} : \operatorname{Re} w + f(z, \bar{z}) < 0\}.$$

Weak locus. Since f contains no mixed x, y-terms, its complex Hessian is given by

$$H_f(z, \bar{z}) = \frac{1}{4}\left(D_x^2 \Phi(x) + D_y^2 |y|^{2m}\right).$$

In view of Lemma 2.7, it suffices to determine the zero set of $\det H_f$. Equivalently, we shall prove that for $(x, y) \in U \times \mathbb{R}^{n-1}$,

$$\det\left(D_x^2 \Phi(x) + D_y^2 |y|^{2m}\right) = 0 \qquad \Longleftrightarrow \qquad y = 0, \quad x_{n-1} = 0, \quad \psi(\hat{x}) = 0. \tag{6.2}$$

The implication from right to left is straightforward when $m \geq 2$ from the construction of f. Indeed, if $y = 0$, then $D_y^2 |y|^{2m} = 0$, and hence by (6.1)

$$\det\left(D_x^2 \Phi(x) + D_y^2 |y|^{2m}\right) = \det D_x^2 \Phi(x) = |x_{n-1}|^{2m} + \psi(\hat{x}),$$

which vanishes when $x_{n-1} = 0$ and $\psi(\hat{x}) = 0$.

Conversely, suppose that the left-hand side of (6.2) vanishes. Note that $D_x^2 \Phi(x) \geq 0$ and $D_y^2 |y|^{2m}$ is positive definite except at $y = 0$. Thus, if $\det\left(D_x^2 \Phi(x) + D_y^2 |y|^{2m}\right) = 0$, then $y = 0$, where $D_y^2 |y|^{2m} = 0$. By (6.1)

$$\det\left(D_x^2 \Phi(x)\right) = |x_{n-1}|^{2m} + \psi(\hat{x}) = 0.$$

But this equality can happen only when $x_{n-1} = \psi(\hat{x}) = 0$. We thus have the desired weak locus.

D'Angelo type. To compute the type at a point $p = ((\hat{x}_p, 0) + i0, w_p) \in W$, where $\hat{x}_p \in E$ and $w_p = -\Phi(\hat{x}_p, 0) + iv_p$, we perform a global biholomorphic coordinate shift $(z, w) \to (\tilde{z}, \tilde{w})$, which sends p to the origin and removes the linear components in the potential:

$$\tilde{z} = z - (\hat{x}_p, 0);$$
$$\tilde{w} = w - w_p + \sum_{j=1}^{n-1} \frac{\partial \Phi}{\partial x_j}(\hat{x}_p, 0)\tilde{z}_j.$$

In these coordinates, the normalized local defining function takes the form

$$\tilde{\rho}(\tilde{z}, \tilde{w}) = \operatorname{Re}\, \tilde{w} + \tilde{\Phi}(\tilde{x}) + |\tilde{y}|^{2m},$$

where

$$\tilde{\Phi}(\tilde{x}) = \Phi(\tilde{x} + (\hat{x}_p, 0)) - \Phi(\hat{x}_p, 0) - \nabla\Phi(\hat{x}_p, 0) \cdot \tilde{x}.$$

Thus $\tilde{\Phi}(0) = \tilde{\Phi}'(0) = 0$. Since Φ is convex, it lies above its supporting hyperplane at $(\hat{x}_p, 0)$. Hence

$$\tilde{\Phi}(\tilde{x}) \geq 0 \tag{6.3}$$

near the origin.

Let $\gamma(\zeta) = (\tilde{z}(\zeta), \tilde{w}(\zeta))$ be a nonconstant holomorphic curve with $\gamma(0) = 0$. Pulling back the normalized local defining function along this curve gives

$$(\tilde{\rho} \circ \gamma)(\zeta) = \operatorname{Re} \tilde{w}(\zeta) + \tilde{\Phi}(\operatorname{Re} \tilde{z}(\zeta)) + |\operatorname{Im} \tilde{z}(\zeta)|^{2m}.$$

First consider $\nu(\tilde{w}(\zeta)) < \nu(\tilde{z}(\zeta))$. Since $\tilde{\Phi}$ vanishes to second order at the origin and $m \geq 2$, the function $\operatorname{Re} \tilde{w}(\zeta)$ becomes the leading term of $(\tilde{\rho} \circ \gamma)(\zeta)$. Hence $\nu(\tilde{\rho} \circ r) = \nu(\tilde{w}(\zeta))$, and

$$\frac{\nu(\tilde{\rho} \circ \gamma)}{\nu(\gamma)} = 1.$$

We now assume $\nu(\tilde{w}(\zeta)) \geq \nu(\tilde{z}(\zeta))$. Let P_d be the first nonzero homogeneous term in the Taylor expansion of $P(\zeta) := \tilde{\Phi}(\operatorname{Re}(\tilde{z}(\zeta))) + |\operatorname{Im}(\tilde{z}(\zeta))|^{2m}$ at 0. Since $\tilde{\Phi} \geq 0$ by (6.3), the function P is nonnegative. Moreover, the term $|\operatorname{Im} \tilde{z}(\zeta)|^{2m}$ has vanishing order $2m\nu(\tilde{z}(\zeta))$. Hence P_d is also nonnegative, with degree

$$d \leq 2m\nu(\tilde{z}(\zeta)).$$

Furthermore, since $P_d(0) = 0$, the nonnegative function P_d cannot be harmonic by the maximum principle. On the other hand, $\operatorname{Re} \tilde{w}(\zeta)$ is harmonic. Thus it cannot cancel the nonharmonic polynomial P_d. Consequently,

$$\nu\big(\tilde{\rho} \circ \gamma\big) \leq d \leq 2m\nu(\tilde{z}(\zeta)),$$

and

$$\frac{\nu(\tilde{\rho} \circ \gamma)}{\nu(\gamma)} \leq \frac{2m\nu(\tilde{z}(\zeta))}{\nu(\tilde{z}(\zeta))} = 2m.$$

This completes the proof. $\square$

7 The Regularized Maximum Function

To construct desired smoothly bounded pseudoconvex domains, we shall truncate the unbounded domains or extend the local domains obtained in the previous sections using ellipsoidal barriers. This requires patching local defining functions with global bounding structures and smoothing the resulting corners. To carry out this procedure without introducing negative eigenvalues that could destroy pseudoconvexity, we shall use the following smooth convex regularized maximum function.

Lemma 7.1. *For any $\epsilon > 0$, there exists a smooth function $M_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ such that the following hold.*

1. *Exactness:*

$$M_\epsilon(s, t) = \max\{s, t\} \qquad \text{if} \qquad |s - t| \geq \epsilon.$$

2. Lower and upper bounds:

$$\max\{s,t\} \leq M_\epsilon(s,t) \leq \frac{\epsilon}{2} + \max\{s,t\} \qquad \textit{for} \qquad (s,t) \in \mathbb{R}^2.$$

3. Zero-set: If $M_\epsilon(s,t) = 0$ and $|s-t| \leq \epsilon$, then

$$-\frac{3\epsilon}{2} \leq s \leq 0 \qquad \textit{and} \qquad -\frac{3\epsilon}{2} \leq t \leq 0.$$

4. Monotonicity:

$$\frac{\partial M_\epsilon}{\partial s}(s,t) \geq 0, \quad \frac{\partial M_\epsilon}{\partial t}(s,t) \geq 0, \quad \frac{\partial M_\epsilon}{\partial s}(s,t) + \frac{\partial M_\epsilon}{\partial t}(s,t) = 1 \qquad \textit{for} \qquad (s,t) \in \mathbb{R}^2,$$

and

$$\frac{\partial M_\epsilon}{\partial s}(s,t) > 0, \quad \frac{\partial M_\epsilon}{\partial t}(s,t) > 0 \qquad \textit{if} \qquad |s-t| < \epsilon.$$

5. Convexity: The real Hessian matrix

$$D^2 M_\epsilon(s,t) = \phi_\epsilon(s-t) \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \textit{for} \qquad (s,t) \in \mathbb{R}^2$$

for some nonnegative smooth function $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ with support on $[-\epsilon, \epsilon]$. In particular, the matrix $D^2 M_\epsilon$ is positive semi-definite on $\mathbb{R}^2$.

Proof. We begin with the classical algebraic identity for the maximum function:

$$\max\{s,t\} = \frac{s+t}{2} + \frac{|s-t|}{2}.$$

Let $\phi_\epsilon \in C_c^\infty(\mathbb{R})$ be a standard symmetric mollifier such that $\phi_\epsilon(y) \geq 0$, $\mathrm{supp}(\phi_\epsilon) \subset [-\epsilon, \epsilon]$, $\phi_\epsilon(-y) = \phi_\epsilon(y)$, and

$$\int_{-\epsilon}^{\epsilon} \phi_\epsilon(y)\, dy = 1. \tag{7.1}$$

We define the regularized absolute value function $\chi_\epsilon : \mathbb{R} \to \mathbb{R}$ via convolution

$$\chi_\epsilon(x) = (|x| * \phi_\epsilon)(x) = \int_{-\epsilon}^{\epsilon} |x-y| \phi_\epsilon(y)\, dy.$$

The regularized maximum function is then defined as

$$M_\epsilon(s,t) := \frac{s+t}{2} + \frac{1}{2}\chi_\epsilon(s-t). \tag{7.2}$$

Clearly, the function $M_\epsilon \in C^\infty(\mathbb{R}^2)$.

Exactness. By the definition of M_ϵ, it suffices to show that $\chi_\epsilon(x) = |x|$ outside $(-\epsilon, \epsilon)$. For any $x \geq \epsilon$ and $y \in [-\epsilon, \epsilon]$ within the support of ϕ_ϵ, we have $x - y \geq 0$. Thus

$$\chi_\epsilon(x) = \int_{-\epsilon}^{\epsilon} (x-y)\phi_\epsilon(y)\, dy = x \int_{-\epsilon}^{\epsilon} \phi_\epsilon(y)\, dy - \int_{-\epsilon}^{\epsilon} y\phi_\epsilon(y)\, dy = x = |x|. \tag{7.3}$$

Here we have used the evenness of ϕ_ϵ and the identity (7.1). The case when $x \le -\epsilon$ is proved similarly by symmetry.

Lower and upper bounds. Since $\phi_\epsilon(y)dy$ is a probability measure, one has

$$\chi_\epsilon(x) = \int_{-\epsilon}^{\epsilon} |x-y|\phi_\epsilon(y)\,dy \ge \left|\int_{-\epsilon}^{\epsilon} (x-y)\phi_\epsilon(y)\,dy\right| = |x|,$$

where the last equality follows from the same computation as in (7.3). Consequently, by the definition (7.2) of M_ϵ,

$$M_\epsilon(s,t) \ge \frac{s+t}{2} + \frac{1}{2}|s-t| = \max\{s,t\}.$$

The upper bound follows from the fact that

$$\chi_\epsilon(x) \le \int_{-\epsilon}^{\epsilon} |x|\phi_\epsilon(y)\,dy + \int_{-\epsilon}^{\epsilon} |y|\phi_\epsilon(y)\,dy \le |x| + \epsilon,$$

and the definition (7.2).

Zero-set. This is a direct consequence of the lower and upper bounds above. Indeed, if $M_\epsilon(s,t) = 0$, then the bounds of M_ϵ imply that

$$0 \ge \max\{s,t\} \ge -\frac{\epsilon}{2}.$$

Suppose first that $\max\{s,t\} = s$. Then

$$0 \ge s \ge -\frac{\epsilon}{2}.$$

On the other hand, by assumption $|s-t| \le \epsilon$, we have

$$s - \epsilon \le t \le s.$$

It then follows that

$$0 \ge s \ge t \ge s - \epsilon \ge -\frac{3\epsilon}{2}.$$

The case when $\max\{s,t\} = t$ is proved in the same way.

Monotonicity. Differentiating $M_\epsilon(s,t)$ with respect to s and t leads to

$$\frac{\partial M_\epsilon}{\partial s} = \frac{1}{2} + \frac{1}{2}\chi_\epsilon'(s-t), \quad \frac{\partial M_\epsilon}{\partial t} = \frac{1}{2} - \frac{1}{2}\chi_\epsilon'(s-t). \tag{7.4}$$

Thus, we only need to verify that

$$-1 \le \chi_\epsilon'(x) \le 1, \quad x \in \mathbb{R}. \tag{7.5}$$

and

$$-1 < \chi_\epsilon'(x) < 1, \quad |x| < \epsilon. \tag{7.6}$$

In fact, differentiating the convolution expression for $\chi_\epsilon(x)$ yields

$$\chi_\epsilon'(x) = \frac{d}{dx}\int_{-\epsilon}^{\epsilon} |x-y|\phi_\epsilon(y)\,dy = \int_{-\epsilon}^{\epsilon} \operatorname{sgn}(x-y)\phi_\epsilon(y)\,dy.$$

Since $-1 \le \operatorname{sgn}(x-y) \le 1$ and $\phi_\epsilon(y) \ge 0$, (7.5) then follows from (7.1). Moreover, (7.6) holds for $|x| < \epsilon$ because $\operatorname{sgn}(x-y)$ is neither identically 1 nor identically -1 on the support of ϕ_ϵ.

Convexity. Recalling that the distributional derivative of $\operatorname{sgn}(x)$ is $2\delta_0(x)$, where δ_0 denotes the Dirac delta function at the origin, we obtain

$$\chi_\epsilon''(x) = \frac{d}{dx}(\operatorname{sgn} * \phi_\epsilon)(x) = (2\delta_0 * \phi_\epsilon)(x) = 2\phi_\epsilon(x), \quad x \in \mathbb{R}.$$

Since $\phi_\epsilon(x) \ge 0$, we have $\chi_\epsilon''(x) \ge 0$. Applying the chain rule to $M_\epsilon(s,t)$, the real Hessian matrix $D^2 M_\epsilon$ takes the explicit form

$$D^2 M_\epsilon(s,t) = \begin{pmatrix} \frac{\partial^2 M_\epsilon}{\partial s^2} & \frac{\partial^2 M_\epsilon}{\partial s \partial t} \\ \frac{\partial^2 M_\epsilon}{\partial t \partial s} & \frac{\partial^2 M_\epsilon}{\partial t^2} \end{pmatrix} = \frac{1}{2}\chi_\epsilon''(s-t)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \phi_\epsilon(s-t)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Since $\phi_\epsilon(s-t) \ge 0$ and the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semi-definite, the Hessian matrix is everywhere positive semi-definite, proving that M_ϵ is convex. □

8 Truncating unbounded models in dimension two

In this section, we construct smoothly bounded convex domains of finite D'Angelo type in $\mathbb{C}^2$ with prescribed weakly pseudoconvex loci. The construction is obtained by truncating the rigid models from Proposition 4.1 with sphere barrier functions of sufficiently large radius and then smoothing the resulting corners using the regularized maximum function. This will prove Theorem 1.1.

Let ρ_{loc} denote the defining function constructed in Proposition 4.1. Namely,

$$\rho_{\text{loc}}(z,w) = \operatorname{Re} w + y^{2m} + \Phi(x),$$

where Φ is given by (4.3). We denote the corresponding rigid cylinder domain by Ω_{loc}. To construct a bounded pseudoconvex domain with the prescribed weak locus, we truncate Ω_{loc} with a strictly convex global ball defined by

$$\rho_{\text{out}}(z,w) = |z|^2 + |w|^2 - K$$

using the regularized maximum function $M_\epsilon(s,t) = \frac{s+t}{2} + \frac{1}{2}\chi_\epsilon(s-t)$ in Section 7. More precisely, we set

$$\rho(z,w) = M_\epsilon\left(\rho_{\text{loc}}(z,w), \rho_{\text{out}}(z,w)\right). \tag{8.1}$$

Here, the positive constants ϵ and K are chosen such that

$$0 < \epsilon < 1 \qquad \text{and} \qquad K > a^2 + \sup_{x \in E} \Phi(x)^2 + 4, \tag{8.2}$$

where the constant $a > 0$ is such that the compact set $E \subset [-a, a]$.

Boundedness. Making use of the lower bound of the regularized maximum function in Lemma 7.1, we have

$$\rho(z,w) = M_\epsilon(\rho_{\mathrm{loc}}(z,w), \rho_{\mathrm{out}}(z,w)) \geq \rho_{\mathrm{out}}(z,w).$$

Therefore $\Omega = \{(z,w) \in \mathbb{C}^2 : \rho(z,w) < 0\}$ is a subset of $\{(z,w) \in \mathbb{C}^2 : \rho_{\mathrm{out}}(z,w) = |z|^2 + |w|^2 - K < 0\}$, which is bounded.

Smoothness. The smoothness of ρ follows from the smoothness of ρ_{loc}, ρ_{out}, and the regularized maximum function M_ϵ. To see that $b\Omega$ is smooth, we further show that $\nabla\rho \neq 0$ on $b\Omega$. By the chain rule,

$$\nabla\rho = \alpha\nabla\rho_{\mathrm{loc}} + \beta\nabla\rho_{\mathrm{out}},$$

with $\alpha := \frac{\partial M_\epsilon}{\partial s} \geq 0$, $\beta := \frac{\partial M_\epsilon}{\partial t} \geq 0$ and $\alpha + \beta = 1$ by monotonicity in Lemma 7.1. Suppose for contradiction that $\nabla\rho = 0$ at some point $p \in b\Omega$.

If $\alpha = 0$, then $\beta = 1$, and hence $\nabla\rho_{\mathrm{out}} = 0$. This forces $p = (0,0)$. However, by the choice of ϵ and K in (8.2),

$$\rho_{\mathrm{out}}(0) = -K < -\epsilon - \Phi(0)^2 - 1 \leq -\epsilon - 2|\Phi(0)| \leq -\epsilon + \Phi(0) = -\epsilon + \rho_{\mathrm{loc}}(0),$$

Thus, by the exactness property in Lemma 7.1, we have $\rho = \rho_{\mathrm{loc}}$ near $(0,0)$ where the gradient is nonzero. This contradicts $\nabla\rho(p) = 0$. On the other hand, if $\beta = 0$, then $\alpha = 1$, and $\nabla\rho_{\mathrm{loc}} = 0$, which is impossible because $\frac{\partial\rho_{\mathrm{loc}}}{\partial w} = \frac{1}{2}$. Therefore $\alpha > 0$ and $\beta > 0$, so the point lies in the transition region $|\rho_{\mathrm{loc}} - \rho_{\mathrm{out}}| < \epsilon$.

The equation $\nabla\rho = 0$ gives, componentwise,

$$\begin{aligned}
&\alpha + 2\beta u = 0;\\
&\beta v = 0;\\
&\alpha\Phi'(x) + 2\beta x = 0;\\
&\alpha m y^{2m-1} + \beta y = 0.
\end{aligned}$$

Since $\alpha, \beta, m > 0$, the second and the fourth equations give

$$v = 0, \qquad y = 0.$$

The first equation gives

$$u = -\frac{\alpha}{2\beta} < 0.$$

Multiplying the third equation by x, we obtain $\alpha x\Phi'(x) + 2\beta x^2 = 0$. Since $x\Phi'(x) \geq 0$ by (4.4), we get

$$x = 0.$$

Thus the hypothetical singular boundary point must be of the form

$$(0, 0, u, 0), \qquad u < 0.$$

At such a point,

$$\rho_{\mathrm{loc}} = u, \qquad \rho_{\mathrm{out}} = u^2 - K.$$

Making use of the zero-set property in Lemma 7.1, we further have

$$-\frac{3\epsilon}{2} \le u \le 0 \qquad \text{and} \qquad -\frac{3\epsilon}{2} \le u^2 - K \le 0.$$

In particular, since $0 < \epsilon < 1$,

$$K \le \frac{3\epsilon}{2} + u^2 \le \frac{3\epsilon}{2} + \frac{9\epsilon^2}{4} < 4,$$

which contradicts the choice of K. Hence no such point exists, and $b\Omega$ is smooth.

Convexity. Since ρ_{loc} and ρ_{out} are both convex, the convexity of ρ is a direct consequence of Lemma A.2. Hence the sub-level set Ω of ρ is convex. In particular, Ω is pseudoconvex.

Weak locus. To investigate the weak locus of the bounded domain Ω, we partition its total physical boundary $b\Omega$ into three mutually exclusive geometric zones determined strictly by the sign and magnitude of $\rho_{\text{loc}}(z,w) - \rho_{\text{out}}(z,w)$.

Zone 1: The far-field exterior spherical shell.

$$\mathcal{Z}_1 = \{(z,w) \in b\Omega : \rho_{\text{loc}}(z,w) - \rho_{\text{out}}(z,w) \le -\epsilon\}.$$

By the exactness property of the regularized maximum operator M_ϵ established in Lemma 7.1, the outer barrier function ρ_{out} dominates the local cylinder model by the ϵ margin and hence

$$\rho = \rho_{\text{out}} \qquad \text{on} \;\; \mathcal{Z}_1.$$

The Levi form matches the sphere exactly: $L_\rho(Z) = L_{\rho_{\text{out}}}(Z) > 0$. This region is strictly pseudoconvex and contains zero weakly pseudoconvex points.

Zone 2: The non-uniform transition band.

$$\mathcal{Z}_2 = \{(z,w) \in b\Omega : |\rho_{\text{loc}}(z,w) - \rho_{\text{out}}(z,w)| < \epsilon\}.$$

Inside $\mathcal{Z}_2$, the smoothing operator M_ϵ blends the local cylinder function and the outer barrier function within an ϵ-distance of each other, and $\alpha = \frac{\partial M_\epsilon}{\partial s} > 0$, $\beta = \frac{\partial M_\epsilon}{\partial t} > 0$. For $\xi = (\xi_1, \xi_2) \in T_p^{1,0}(b\Omega)$, the Levi form of the domain is

$$\begin{aligned}
L_\rho(\xi,\bar\xi) &= \alpha L_{\rho_{\text{loc}}}(\xi,\bar\xi) + \beta L_{\rho_{\text{out}}}(\xi,\bar\xi) \\
&\qquad + (M_\epsilon)_{ss}(s,t)\,|\partial s(\xi)|^2 + 2\,\mathrm{Re}\left((M_\epsilon)_{st}(s,t)\,\partial s(\xi)\overline{\partial t(\xi)}\right) + (M_\epsilon)_{tt}(s,t)\,|\partial t(\xi)|^2 \\
&= \alpha L_{\rho_{\text{loc}}}(\xi,\bar\xi) + \beta L_{\rho_{\text{out}}}(\xi,\bar\xi) + \begin{pmatrix} \overline{\partial s(\xi)} & \overline{\partial t(\xi)} \end{pmatrix} D^2 M_\epsilon(s,t) \begin{pmatrix} \partial s(\xi) \\ \partial t(\xi) \end{pmatrix}.
\end{aligned} \tag{8.3}$$

where $\partial s(\xi) = \sum_{j=1}^2 s_j \xi_j$ and $\partial t(\xi) = \sum_{j=1}^2 t_j \xi_j$ when viewing ξ as a vector field. Since $D^2 M_\epsilon$ is positive semi-definite, the last term is nonnegative. Thus the total complex Levi form of the domain satisfies

$$L_\rho \ge \beta L_{\rho_{\text{out}}} > 0.$$

The transition layer is strictly pseudoconvex and creates zero weakly pseudoconvex points.

Zone 3: The deep interior local floor.

$$\mathcal{Z}_3 = \{(z,w) \in b\Omega : \rho_{\text{loc}}(z,w) - \rho_{\text{out}}(z,w) \geq \epsilon\}.$$

In this region, the local cylinder model dominates the outer spherical barrier function by an explicit thickness margin of ϵ. By Lemma 7.1, the regularized maximum reduces to

$$\rho = \rho_{\text{loc}} \qquad \text{on} \quad \mathcal{Z}_3.$$

Therefore the weak locus W of Ω satisfies

$$W = W_{\text{loc}} \cap \mathcal{Z}_3,$$

where W_{loc} denotes the weak locus of ρ_{loc}.

To show that Ω has the desired weak locus, it suffices to verify that every point of the form $p = (x_p + i0, -\Phi(x_p) + iv_p) \in W_{\text{loc}}$, where $x_p \in E$ and $|v_p| < 1$, satisfies $\rho_{\text{loc}} - \rho_{\text{out}} \geq \epsilon$. Indeed, for each such point, we have

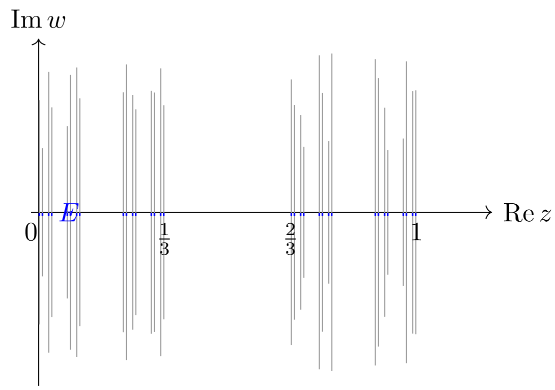
$$\rho_{\text{loc}}(p) - \rho_{\text{out}}(p) = 0 - \big(|x_p|^2 + |\Phi(x_p)|^2 + |v_p|^2 - K\big) = K - |x_p|^2 - \Phi(x_p)^2 - |v_p|^2.$$

By the choice of K in (8.2),

$$K - |x_p|^2 - \Phi(x_p)^2 - |v_p|^2 > K - a^2 - \sup_{x \in E} \Phi(x)^2 - 1 > \epsilon.$$

Namely, $p \in \mathcal{Z}_3$. Consequently, the weakly pseudoconvex locus of Ω_{loc} in the region $|\operatorname{Im} w| < 1$ remains unchanged in Ω. Moreover, the corresponding D'Angelo type at these points is carried over to Ω unchanged.

The following figure schematically illustrates the weakly pseudoconvex locus in Theorem 1.1 when E is the classical middle-thirds Cantor set on $[0,1]$. The weak locus consists of a family of one-dimensional fibers emanating from E in the Reeb direction, forming a "Cantor forest" of vertical segments rooted at E. The finitely many segments displayed schematically represent the uncountable family of fibers lying over E.



Weak locus in Theorem 1.1
$W = \{(z,w) \in b\Omega : \operatorname{Re} z \in E,\ \operatorname{Im} z = 0\} \cap \overline{\mathcal{U}}$

Figure 1: Schematic representation of the weakly pseudoconvex locus in the $(\operatorname{Re} z, \operatorname{Im} w)$-plane, corresponding to the complex-tangential and Reeb directions, respectively. The condition $\operatorname{Im} z = 0$ is suppressed, while $\operatorname{Re} w$ is determined by the defining equation in Proposition 4.1, and the extent in the $\operatorname{Im} w$-direction is restricted by the neighborhood $\mathcal{U}$. The base set $E \subset \mathbb{R}$ in blue is represented by the middle-thirds Cantor set on $[0,1]$, with five iterations shown.

9 Constructing convex models in dimension two, continued

Given a compact set $E \subset \mathbb{R}$, we construct in this section a smoothly bounded convex domain in $\mathbb{C}^2$ whose weak locus is diffeomorphic to E. This will prove Theorem 1.2. Write $(z,w) = (x+iy, u+iv) \in \mathbb{C}^2$. After a translation in the x-variable, we may assume without loss of generality that $0 \notin E$.

To eliminate the tree-like components over the base points in E in Theorem 1.1, we modify the local rigid model by adding a controlling term in the v direction. More precisely, for fixed integer $N \geq m \geq 2$, we define the local model by

$$\rho_{\text{loc}}(z,w) = u + v^{2N} + y^{2m} + \Phi(x),$$

where Φ is defined as in (4.3). The strictly convex global barrier and the patched defining function stays the same:

$$\rho_{\text{out}}(z,w) = x^2 + y^2 + u^2 + v^2 - K,$$

where $K > 0$ is chosen sufficiently large as in (8.2), and

$$\rho(z,w) = M_\epsilon\big(\rho_{\text{loc}}(z,w), \rho_{\text{out}}(z,w)\big).$$

Set

$$\Omega = \{(z,w) \in \mathbb{C}^2 : \rho(z,w) < 0\}.$$

The boundedness of Ω follows from $\rho_{\text{out}} \leq \rho < 0$. Since both ρ_{loc} and ρ_{out} are convex, the convexity of Ω then follows from Lemma A.2. In particular, Ω is pseudoconvex.

Smoothness. Suppose that

$$\nabla\rho = \alpha\nabla\rho_{\text{loc}} + \beta\nabla\rho_{\text{out}}$$

is zero at some point of $b\Omega$. This can only happen when both $\alpha > 0$ and $\beta > 0$ as argued in Section 8, so the point lies in the transition region $|\rho_{\text{loc}} - \rho_{\text{out}}| < \epsilon$. In this case, by inspecting the components of $\nabla\rho$, one has

$$\begin{aligned}
&\alpha\, N v^{2N-1} + \beta v = 0;\\
&\alpha\, m y^{2m-1} + \beta y = 0;\\
&\alpha\, \Phi'(x) + 2\beta x = 0;\\
&\alpha + 2\beta u = 0.
\end{aligned}$$

With a similar argument as in Section 8, one can obtain that any hypothetical singular boundary point must be of the form

$$(0, 0, u, 0), \qquad u < 0.$$

At such a point,

$$\rho_{\text{loc}} = u, \qquad \rho_{\text{out}} = u^2 - K.$$

This is impossible since K is chosen sufficiently large as in (8.2). Hence $b\Omega$ is smooth.

Weak locus. The Levi form takes a similar form as in (8.3). Hence the weakly pseudoconvex points lie in the region where $\beta = 0$ and $\alpha = 1$, where $\rho_{\text{loc}} - \rho_{\text{out}} \geq \epsilon$, and

$$\rho = \rho_{\text{loc}}.$$

On this region, the weakly pseudoconvex locus is determined by the Levi form of ρ_{loc}:

$$L(\xi, \bar{\xi}) = \frac{1}{4}|\xi_1|^2 \left(\psi(x) + 2m(2m-1)y^{2m-2}\right) + \frac{1}{2}|\xi_2|^2\, N(2N-1)v^{2N-2} \tag{9.1}$$

for every nonzero vector $\xi = (\xi_1, \xi_2) \in T_p^{1,0}(b\Omega)$ that satisfies

$$\xi_1(\Phi'(x) - 2imy^{2m-1}) + \xi_2(1 - 2iNv^{2N-1}) = 0. \tag{9.2}$$

We first note that

$$\xi_1 \neq 0.$$

Otherwise, since the coefficient $1 - 2iNv^{2N-1}$ is never zero in (9.2), it follows that $\xi_2 = 0$. This contradicts the nonzero assumption of ξ.

Since the Levi form (9.1) is a sum of nonnegative terms, its vanishing forces each term in the sum to vanish. Therefore, using the construction of ψ and the fact that $\xi_1 \neq 0$, we obtain

$$x \in E, \qquad y = 0, \qquad \xi_2 v = 0.$$

On the other hand, since the nonnegative function ψ is strictly positive off E and $0 \notin E$,

$$\Phi'(x) = \int_0^x \psi(t)dt \neq 0 \quad \text{for all} \quad x \in E.$$

This, together with (9.2), implies $\xi_2 \neq 0$. Thus

$$v = 0.$$

Hence, the weakly pseudoconvex locus of the local model is precisely the prescribed set. By the choice of K in (8.2), together with a similar argument at the end of the previous section for zone 3, these weakly pseudoconvex points are carried over from Ω_{loc} to Ω. The desired weak locus is thus verified.

D'Angelo type. Fix a point $p = (z_p, w_p) = (x_p + i0, -\Phi(x_p) + i0) \in W$ with $x_p \in E$. We use the biholomorphic change of coordinates

$$\begin{aligned} \tilde{z} &= z - x_p; \\ \widetilde{w} &= w - w_p + \Phi'(x_p)(z - x_p), \end{aligned}$$

which sends p to the origin. In these coordinates, the defining function has the local form

$$\widetilde{\rho}(\tilde{z}, \widetilde{w}) = \operatorname{Re}\widetilde{w} + \left(\operatorname{Im}\widetilde{w} - \Phi'(x_p)\operatorname{Im}\tilde{z}\right)^{2N} + (\operatorname{Im}\tilde{z})^{2m} + R(\operatorname{Re}\tilde{z}),$$

where R is flat at the origin.

Let $\gamma(\zeta) = (\tilde{z}(\zeta), \widetilde{w}(\zeta))$ be a nonconstant holomorphic curve with $\gamma(0) = 0$. Pulling back $\widetilde{\rho}$ along γ, we obtain

$$(\widetilde{\rho} \circ \gamma)(\zeta) = \operatorname{Re}\widetilde{w}(\zeta) + \left(\operatorname{Im}\widetilde{w}(\zeta) - \Phi'(x_p)\operatorname{Im}\tilde{z}(\zeta)\right)^{2N} + (\operatorname{Im}\tilde{z}(\zeta))^{2m} + R(\operatorname{Re}\tilde{z}(\zeta)).$$

First suppose that $\nu(z(\zeta)) \geq \nu(w(\zeta))$. Then $\operatorname{Re}\widetilde{w}(\zeta)$ has vanishing order $\nu(w(\zeta))$, while all other terms have order strictly larger than $\nu(w(\zeta))$. Hence $\nu(\widetilde{\rho} \circ \gamma) = \nu(w(\zeta))$, and

$$\frac{\nu(\widetilde{\rho} \circ \gamma)}{\nu(\gamma)} = 1.$$

Now suppose that $\nu(z(\zeta)) < \nu(w(\zeta))$. Then $\nu(\gamma) = \nu(z(\zeta))$. Moreover, $(\operatorname{Im}\tilde{z}(\zeta))^{2m}$ has vanishing order $2m\nu(z(\zeta))$ and $R(\operatorname{Re}\tilde{z}(\zeta))$ vanishes to infinite order. Similarly as in the proof of Proposition 6.2, let P_d be the first nonzero homogeneous term in the Taylor expansion of $\left(\operatorname{Im}\widetilde{w}(\zeta) - \Phi'(x_p)\operatorname{Im}\tilde{z}(\zeta)\right)^{2N} + (\operatorname{Im}\tilde{z}(\zeta))^{2m}$. Then P_d is nonnegative and not constantly zero, with degree

$$d \leq 2m\nu(z(\zeta)).$$

Since $P_d(0) = 0$, the maximum principle implies that P_d cannot be harmonic. On the other hand, every homogeneous term of $\operatorname{Re}\widetilde{w}(\zeta)$ is harmonic. Thus $\operatorname{Re}\widetilde{w}(\zeta)$ cannot cancel P_d. Therefore

$$\nu(\widetilde{\rho} \circ \gamma) \leq d \leq 2m\nu(z(\zeta)).$$

and

$$\frac{\nu(\widetilde{\rho} \circ \gamma)}{\nu(\gamma)} \leq 2m.$$

Moreover, this bound is sharp by letting the curve be $\gamma(\zeta) = (\zeta, 0)$, which gives

$$\nu(\widetilde{\rho} \circ \gamma) = 2m.$$

Hence the D'Angelo type along the prescribed weak locus is exactly $2m$.

10 Extending local models in higher dimensions

In this section, we prove Theorems 1.3 and 1.4 by smoothly extending the local hypersurfaces obtained in Propositions 5.2 and 6.2 using an outer ellipsoidal barrier and the regularized maximum function.

Fix $x_0 \in E \subset \mathbb{R}^{n-1}$ and $m \geq 2$. Let ρ_{loc} be the local defining function constructed in Proposition 5.2, defined on $U \times \mathbb{C}$ and denote the corresponding local hypersurface by M_{loc}. Namely,

$$\rho_{\text{loc}}(z, w) = \operatorname{Re} w + f(z, \bar{z})$$

with f satisfying (5.1). Without loss of generality, we may assume that the neighborhood $U := B_\delta(x_0 + i0)$, the ball centered at $x_0 + i0$ with radius $\delta > 0$ in $\mathbb{C}^{n-1}$. For $\epsilon > 0$, we shall patch the local defining function with the strictly convex ellipsoidal barrier:

$$\rho_{\text{out}}(z, w) = A|z - x_0|^2 + |w|^2 - K,$$

where A and K are constants satisfying

$$A > \max\left\{ \frac{32(\sup_{z\in U} |f(z, \bar{z})|^2 + \epsilon)}{\delta^2}, \frac{4(\sup_{z\in U} |f(z, \bar{z})| + 2\epsilon) \sup_{z\in U} |\nabla f(z, \bar{z})|}{\delta}, 1 \right\} \tag{10.1}$$

and

$$\frac{A\delta^2}{4} > K > \frac{A\delta^2}{8} + \sup_{z\in U} |f(z, \bar{z})|^2 + \epsilon. \tag{10.2}$$

Such a choice of K is possible by (10.1). In particular, one has

$$\frac{A\delta^2}{4} > K > \frac{A\delta^2}{8} \qquad \text{and} \qquad 0 < \epsilon < \frac{A\delta^2}{32}. \tag{10.3}$$

As will be seen later, the upper bound of K in (10.2) is to confine the desired global domain within a cylinder where ρ_{loc} is defined, and the lower bound there is to preserve the weak locus of M_{loc} near $(x_0 + i0, -f(x_0, x_0) + i0)$.

We define, on $U \times \mathbb{C}$, our defining function again via the regularized maximum function M_ϵ:

$$\rho(z, w) = M_\epsilon(\rho_{\text{loc}}(z, w), \rho_{\text{out}}(z, w)),$$

and set

$$\Omega = \{(z, w) \in U \times \mathbb{C} : \rho(z, w) < 0\}.$$

We claim that the set Ω is compactly contained in the region $U \times \mathbb{C}$, where ρ_{loc} is well defined. Indeed, by Lemma 7.1, we have

$$\rho_{\text{out}}(z, w) \leq \rho(z, w) \leq 0 \qquad \text{on} \qquad \overline{\Omega}.$$

Since $\rho_{\text{out}}(z, w) = A|z - x_0|^2 + |w|^2 - K$, it follows that

$$A|z - x_0|^2 + |w|^2 \leq K.$$

In particular,

$$|z - x_0| \le \sqrt{\frac{K}{A}} < \frac{\delta}{2} \qquad \text{and} \qquad |w| \le \sqrt{K}.$$

by the choice of K in (10.2). Therefore $z \in B_{\delta/2}(x_0 + i0) \Subset B_\delta(x_0 + i0) = U$. The claim is thus proved. Consequently, $\overline{\Omega}$ cannot meet the boundary of the local coordinate cylinder, and Ω is a well-defined bounded domain in $\mathbb{C}^n$.

To see $b\Omega$ is smooth, we need to verify that $\nabla\rho$ is nonvanishing on $\rho = 0$. Again, since

$$\nabla\rho = \alpha\nabla\rho_{\text{loc}} + \beta\nabla\rho_{\text{out}},$$

with $\alpha \ge 0, \beta \ge 0$ and $\alpha + \beta = 1$, it suffices to only consider the region when both $\alpha > 0$ and $\beta > 0$, namely, the transition region $|\rho_{\text{loc}} - \rho_{\text{out}}| \le \epsilon$. On this region, if $\rho = 0$, then it follows from the zero-set property in Lemma 7.1 that

$$|\rho_{\text{loc}}| < 2\epsilon, \qquad |\rho_{\text{out}}| < 2\epsilon. \tag{10.4}$$

Let

$$S = \sup_{z \in U} |f(z, \bar z)|, \qquad L = \sup_{z \in U} |\nabla f(z, \bar z)|.$$

By the bounds for ρ_{loc} and ρ_{out} in (10.4), together with their explicit expressions, we obtain

$$|u| \le S + 2\epsilon, \tag{10.5}$$

and

$$A|z - x_0|^2 + |w|^2 - K \ge -2\epsilon.$$

Using the inequalities (10.3) for K and ϵ , we obtain

$$A|z - x_0|^2 + v^2 \ge K - 2\epsilon > \frac{A\delta^2}{16}. \tag{10.6}$$

Suppose, for contradiction, that $\nabla\rho = 0$ at a point on $b\Omega$. Since $\beta > 0$ and the v-component of $\nabla\rho = 0$ gives $2\beta v = 0$, we have

$$v = 0.$$

Plugging the above into (10.6) further yields

$$|z - x_0| > \frac{\delta}{4}. \tag{10.7}$$

The u-component of $\nabla\rho = 0$ gives

$$\alpha + 2\beta u = 0.$$

Hence by (10.5)

$$\alpha \le 2\beta|u| \le 2\beta(S + 2\epsilon). \tag{10.8}$$

The z-components of $\nabla\rho = 0$ give

$$\alpha\nabla f(z, \bar z) + 2\beta A(z - x_0) = 0.$$

Taking norms and dividing by 2β, we get from (10.8) that

$$A|z - x_0| \le \frac{\alpha}{2\beta} L \le (S + 2\epsilon)L.$$

Combining this further with (10.7), we obtain

$$\frac{A\delta}{4} < (S + 2\epsilon)L.$$

This contradicts with (10.1). Therefore $\nabla\rho \neq 0$ on $b\Omega$, and the boundary $b\Omega$ is smooth.

The verification of pseudoconvexity of the domain Ω is the same as in Section 8 in dimension $n = 2$, and will be omitted. It remains only to determine the weak locus. As before, we divide into three regions according to the value of $\rho_{\text{loc}} - \rho_{\text{out}}$.

Zone 1: The portion dominated by the outer barrier function ρ_{out}:

$$\mathcal{Z}_1 = \{(z, w) \in b\Omega : \rho_{\text{loc}}(z, w) - \rho_{\text{out}}(z, w) \le -\epsilon\},$$

where

$$\rho = \rho_{\text{out}} \qquad \text{on} \quad \mathcal{Z}_1.$$

The Levi form matches the sphere exactly: $L_\rho(Z) = L_{\rho_{\text{out}}}(Z) > 0$. This region is strictly pseudoconvex and contains zero weakly pseudoconvex points.

Zone 2: The transition layer between the local hypersurface and the barrier sphere:

$$\mathcal{Z}_2 = \{(z, w) \in b\Omega : |\rho_{\text{loc}}(z, w) - \rho_{\text{out}}(z, w)| < \epsilon\}$$

Investigating the Levi form with respect to ρ in exactly the same way as in the case $n = 2$ in Section 8, we can verify that the transition layer is strictly pseudoconvex and creates zero weakly pseudoconvex points.

Zone 3: The portion dominated by ρ_{loc}:

$$\mathcal{Z}_3 = \{(z, w) \in b\Omega : \rho_{\text{loc}}(z, w) - \rho_{\text{out}}(z, w) \ge \epsilon\},$$

where

$$\rho = \rho_{\text{loc}} \qquad \text{on} \quad \mathcal{Z}_3.$$

Therefore the weak locus W of Ω satisfies

$$W = W_{\text{loc}} \cap \mathcal{Z}_3,$$

where W_{loc} denotes the weak locus of M_{loc}.

It then suffices to verify that a neighborhood of $p_0 = (x_0 + i0, -f(x_0, x_0) + i0)$ in M_{loc} is contained in $\mathcal{Z}_3$. Let $p = (z_p, w_p) \in M_{\text{loc}}$, with $z_p \in B_{\frac{\delta}{3}}(x_0 + i0)$, $w_p = -f(z_p, \bar{z}_p) + iv_p$ and $|v_p| < \frac{\delta}{9}$. Then $\rho_{\text{loc}}(p) = 0$. On the other hand, using the lower bound of K in (10.2), we obtain

$$\begin{aligned}\rho_{\text{out}}(p) = A|z_p - x_0|^2 + |w_p|^2 - K &\le \frac{A\delta^2}{9} + \sup_{z\in U} |f(z, \bar{z})|^2 + \frac{\delta^2}{81} - K \\ &< \frac{A\delta^2}{8} + \sup_{z\in U} |f(z, \bar{z})|^2 - K < -\epsilon.\end{aligned}$$

Consequently,

$$\rho_{\mathrm{loc}}(p) - \rho_{\mathrm{out}}(p) > \epsilon.$$

Thus $p \in \mathcal{Z}_3$. Namely, a neighborhood of p_0 in M_{loc} is contained in $\mathcal{Z}_3$, and on this neighborhood the global hypersurface $b\Omega$ agrees with the local hypersurface M_{loc}. Therefore, the corresponding D'Angelo type is carried over to $b\Omega$ unchanged.

For the case $E \subset \mathbb{R}^{n-2}$, Proposition 6.2 gives rise to a local hypersurface M_{loc} defined on $(U + i\mathbb{R}^{n-1}) \times \mathbb{C} \subset \mathbb{C}^n$, where U is a neighborhood of $(\hat{x}_0, 0)$ in $\mathbb{R}^{n-1}$. A careful inspection of the proof shows that the tube neighborhood $U + i\mathbb{R}^{n-1}$ in Proposition 6.2 may be replaced by a ball $B_\delta((\hat{x}_0, 0) + i0) \subset \mathbb{C}^{n-1}$ without affecting the conclusion of the proposition. Therefore the patching argument and verification carried out above for Proposition 5.2 would apply equally to the case $E \subset \mathbb{R}^{n-2}$ in Proposition 6.2, with x_0 replaced by $(\hat{x}_0, 0)$. We therefore omit the details.

11 Further remarks and conjectures

All examples constructed in this paper are of finite D'Angelo type. Moreover, in each of these examples, the prescribed weak locus has zero surface measure on the boundary, and the corresponding Levi determinant vanishes to finite order along this locus. We conclude the paper by posing two natural conjectures concerning the interaction between finite D'Angelo type, the degeneracy of the Levi form, and the size of the weakly pseudoconvex locus.

Conjecture 1: Let Ω be a smoothly bounded pseudoconvex domain of finite D'Angelo type in $\mathbb{C}^n, n \geq 3$. Then the collection of all weakly pseudoconvex points in $b\Omega$ has zero surface measure on $b\Omega$.

Conjecture 2: Let Ω be a smoothly bounded pseudoconvex domain of finite D'Angelo type in $\mathbb{C}^n, n \geq 3$. Then the Levi determinant, regarded as a smooth function on $b\Omega$, is not flat at any weakly pseudoconvex point of $b\Omega$. Equivalently, the Levi determinant does not vanish to infinite order along the tangential directions of $b\Omega$ at any weakly pseudoconvex boundary point.

We begin by proving a relationship between the non-flatness of the Levi determinant and the surface measure of the weakly pseudoconvex locus. In particular, this shows that Conjecture 2 implies Conjecture 1.

Proposition 11.1. *Let $\Omega \subset \mathbb{C}^n$ be a smooth pseudoconvex domain. Assume that, at every weakly pseudoconvex boundary point, the Levi determinant does not vanish to infinite order as a smooth function on $b\Omega$. Then the weakly pseudoconvex locus W of Ω has zero surface measure on $b\Omega$.*

The proposition follows directly from the following elementary lemma, applied in local boundary coordinates to the Levi determinant Λ_ρ.

Lemma 11.2. *Let $h \in C^\infty(V)$, where $V \subset \mathbb{R}^d$ is open. If h does not vanish to infinite order at any point of $h^{-1}(0)$, then $h^{-1}(0)$ has Lebesgue measure zero in $\mathbb{R}^d$.*

Proof. Suppose by contradiction that $E := h^{-1}(0)$ has positive measure. By the Lebesgue's density theorem, there exists a point $x_0 \in E$ which is a density point of E. Equivalently,

$$\lim_{r \to 0} \frac{|B_r(x_0) \setminus E|}{|B_r(x_0)|} = 0.$$

Let k be the least integer such that some derivative of order k of h is nonzero at x_0. Thus the Taylor expansion of h at x_0 has the form

$$h(x_0 + y) = P_k(y) + o(|y|^k),$$

where P_k is a nonzero homogeneous polynomial of degree k. For $r > 0$, set

$$E_r = \{y \in B_1(0)) : x_0 + ry \in E\}.$$

Since x_0 is a density point of E, the measure

$$|B_1(0)) \setminus E_r| \to 0$$

as $r \to 0$. On the other hand,

$$\frac{h(x_0 + ry)}{r^k} \longrightarrow P_k(y)$$

uniformly for $y \in B_1(0)$. Since $h(x_0 + ry) = 0$ on E_r, we obtain

$$\int_{B_1(0)} |P_k(y)|\, dy \le \int_{E_r} \left| P_k(y) - \frac{h(x_0 + ry)}{r^k} \right| dy + \int_{B_1(0) \setminus E_r} |P_k(y)|\, dy.$$

Letting $r \to 0$, the right-hand side tends to zero. Hence $P_k = 0$ almost everywhere on $B_1(0)$, and therefore $P_k \equiv 0$, a contradiction. This proves the lemma. $\square$

For smooth pseudoconvex domains in $\mathbb{C}^2$, Kohn [11, Proposition 2.8] showed that finite commutator type is equivalent to the nonflatness of the Levi determinant along tangential directions. Since finite D'Angelo type and finite commutator type are equivalent in $\mathbb{C}^2$, Conjecture 2 holds when $n = 2$. Consequently, Conjecture 1 also holds in this case by Proposition 11.1. For the reader's convenience, we include the proof below.

We first recall the definition of commutator type. For a smooth real hypersurface $M \subset \mathbb{C}^2$, let L be a local nonvanishing $(1, 0)$-vector field tangent to M near p. For $k \ge 1$, let $\mathcal{L}_k$ denote the collection of all iterated commutators of L and $\overline{L}$ of length at most k. The commutator type of M at p is the smallest integer k such that the values at p of the vector fields in $\mathcal{L}_k$ span the complexified tangent space $\mathbb{C}T_pM$. If no such k exists, the commutator type is defined to be infinite.

We shall also need the following Cartan's formula: if θ is a smooth one-form and X, Y are smooth vector fields, then

$$\theta([X, Y]) = X(\theta(Y)) - Y(\theta(X)) - d\theta(X, Y). \tag{11.1}$$

Proposition 11.3. *Let $\Omega \subset \mathbb{C}^2$ be a smooth pseudoconvex domain of finite D'Angelo type. Then the Levi determinant, regarded as a smooth function on $b\Omega$, does not vanish to infinite order at any weakly pseudoconvex point of $b\Omega$. In particular, the weakly pseudoconvex locus W has zero surface measure on $b\Omega$.*

Proof. The assertion is local on $b\Omega$. Let $p \in b\Omega$, and let ρ be a smooth local defining function near p. Choose a local nonvanishing $(1,0)$-vector field L spanning $T^{1,0}(b\Omega)$ near p. The scalar Levi form is then given by

$$\lambda_\rho = \mathcal{L}_\rho(L, \overline{L}) = \sum_{j,k=1}^{2} \rho_{z_j \bar{z}_k} L_j \overline{L_k}.$$

Since the Levi determinant agrees with λ_ρ up to a nonvanishing scalar factor, it suffices to show that λ_ρ does not vanish to infinite order at p along the tangential directions L and $\overline{L}$.

Let $\theta = \frac{i}{2}(\bar{\partial}\rho - \partial\rho)$ be the real contact form on $b\Omega$. Then $\theta(L) = \theta(\overline{L}) = 0$, and θ does not vanish on the remaining real tangential vector field in $\mathbb{C}T(b\Omega)\big/(T^{1,0}(b\Omega) \oplus T^{0,1}(b\Omega))$. Let C_k be an iterated commutator of L and $\overline{L}$ of length k. We claim that $\theta(C_k)$ is a finite sum of tangential derivatives of λ_ρ of order at most $k-2$, with smooth coefficients. For $k = 2$, we have

$$C_2 = [L, \overline{L}].$$

By Cartan's formula (11.1),

$$\theta([L, \overline{L}]) = -d\theta(L, \overline{L}).$$

Since $d\theta(L, \overline{L})$ is a nonzero smooth multiple of the Levi form, there exists a smooth nonvanishing function a such that

$$\theta(C_2) = a\lambda_\rho.$$

Thus the claim holds for commutators of length 2.

Assume that the claim holds for all commutators of length k. Let X be either L or $\overline{L}$, and consider

$$C_{k+1} = [X, C_k].$$

Again by Cartan's formula and the identity $\theta(X) = 0$, we have

$$\theta(C_{k+1}) = \theta([X, C_k]) = X(\theta(C_k)) - d\theta(X, C_k).$$

Choose a real tangential vector field T such that $\theta(T) = 1$. Then C_k can be written as

$$C_k = \theta(C_k)T + c_1 L + c_2 \overline{L},$$

where c_1 and c_2 are smooth functions. Since $d\theta$ is tensorial, we further obtain

$$\theta(C_{k+1}) = X(\theta(C_k)) - \theta(C_k) d\theta(X, T) - c_1 d\theta(X, L) - c_2 d\theta(X, \overline{L}).$$

By the induction hypothesis, $\theta(C_k)$ involves tangential derivatives of λ_ρ of order at most $k-2$. Hence $X(\theta(C_k))$ involves tangential derivatives of λ_ρ of order at most $k-1$. The term $\theta(C_k) d\theta(X, T)$ involves no derivatives beyond those already present in $\theta(C_k)$, and therefore has order at most $k-2$ in λ_ρ. Finally, the terms $d\theta(X, L)$ and $d\theta(X, \overline{L})$ are either zero or smooth multiples of the Levi form, and hence contain only the factor λ_ρ itself. Altogether, $\theta(C_{k+1})$ involves tangential derivatives of λ_ρ of order at most $k-1$, as desired. This proves the claim.

Consequently, if λ_ρ vanishes to infinite order at p along $b\Omega$, then every finite iterated commutator of L and $\overline{L}$ would have zero component in the missing real tangential direction.

Hence the Lie algebra generated by L and $\overline{L}$ would fail, at every finite order, to generate the full complexified tangent space $\mathbb{C}T_p(b\Omega)$. Thus the commutator type at p would be infinite. However, since finite D'Angelo type is equivalent to finite commutator type in $\mathbb{C}^2$, this contradicts the assumption that p is of finite D'Angelo type. Therefore λ_ρ cannot vanish to infinite order at p. $\square$

A Proof of Lemmas 2.5, A.1 and A.2

We first prove Lemma 2.5, which relates the determinant of the bordered Hessian to the Levi determinant.

Proof of Lemma 2.5. Choose an orthonormal basis $\{e_1, \dots, e_{n-1}\}$ for the $(n-1)$-dimensional complex tangent space $T_p^{1,0}(b\Omega)$. Define the n-th basis vector as the normalized complex normal

$$e_n = \frac{1}{|\partial\rho|}(\bar{\partial}\rho)^T.$$

Construct the $n \times n$ unitary matrix $U = [e_1, e_2, \dots, e_n]$, where e_k are column vectors.

Under the holomorphic coordinate change $z = U\tilde{z}$, we define the localized defining function $\tilde{\rho}(\tilde{z}) = \rho(U\tilde{z})$. In particular,

$$\det(H_\rho^{\mathrm{bd}}) = \det(\tilde{H}_{\tilde{\rho}}^{\mathrm{bd}}),$$

where H_ρ^{bd} and $\tilde{H}_{\tilde{\rho}}^{\mathrm{bd}}$ are the corresponding bordered Hessian matrix with respect to coordinates z and $\tilde{z}$, respectively. By the chain rule,

$$\frac{\partial\tilde{\rho}}{\partial\tilde{z}_k} = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}\frac{\partial z_j}{\partial\tilde{z}_k} = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(e_k)_j = 0,$$

where the last equality is due to the fact that the vector e_k lies in the complex tangent space $T_p^{1,0}(b\Omega)$. For the normal direction $k = n$, substituting the components of e_n yields

$$\frac{\partial\tilde{\rho}}{\partial\tilde{z}_n} = \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} \cdot \frac{1}{|\partial\rho|}\frac{\partial\rho}{\partial\bar{z}_j} = \frac{1}{|\partial\rho|}\sum_{j=1}^{n}\left|\frac{\partial\rho}{\partial z_j}\right|^2 = |\partial\rho|.$$

Thus, in the $\tilde{z}$-coordinates, the complex gradient row vector is exactly

$$\partial_{\tilde{z}}\tilde{\rho} = (0, 0, \dots, 0, |\partial\rho|).$$

On the other hand, by construction and definition, the upper-left $(n-1) \times (n-1)$ block of the transformed Hessian H_ρ is precisely the restricted Hessian $\tilde{H}_\rho$. Therefore, $\tilde{H}_{\tilde{\rho}}^{\mathrm{bd}}$ takes the following form

$$\tilde{H}_{\tilde{\rho}}^{\mathrm{bd}} = \begin{pmatrix} 0 & 0 & \dots & 0 & |\partial\rho| \\ 0 & & & & * \\ \vdots & & \tilde{H}_\rho & & \vdots \\ 0 & & & & * \\ |\partial\rho| & * & \dots & * & * \end{pmatrix}.$$

We then obtain by cofactor expansion that

$$\det(\tilde{H}^{\mathrm{bd}}_{\tilde{\rho}}) = -|\partial\rho|^2 \det(\tilde{H}_\rho) = -|\partial\rho|^2 \prod_{j=1}^{n-1} \lambda_j.$$

The proof is complete. □

The following lemma provides the construction of a locally finite covering for open sets of $\mathbb{R}^n$, as used in Proposition 3.1.

Lemma A.1. *Let U be an open set in $\mathbb{R}^d$. For any $r > 0$, there exists a countable, locally finite collection of open balls with radius less than r that covers U.*

Proof. For each integer $m \geq 1$, define the compact set

$$K_m = \{x \in U : \|x\| \leq m \text{ and } d(x, U^c) \geq 1/m\},$$

where $d(x, A)$ denotes the distance from x to the set A. Then $K_m \subset \mathrm{int}(K_{m+1})$ and $\bigcup_{m=1}^{\infty} K_m = U$. Let $K_0 = \emptyset$ and set

$$V_m = \mathrm{int}(K_{m+1}) \setminus K_{m-1} \quad \text{for } m \geq 1.$$

The collection of open sets $\{V_m\}_{m=1}^{\infty}$ is a locally finite covering of U.

Now, since each $\overline{V_m}$ is compact, $\overline{V_m}$ can be covered by finitely many open balls of radius less than r, each contained in $K_{m+2} \setminus \overline{K_{m-2}}$ (setting $K_{-1} = \emptyset$). Denote this finite collection of balls by $\mathcal{B}_m$. Set

$$\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m.$$

Then $\mathcal{B}$ is a countable collection of open balls with radius less than r, and its union is U. Indeed, every point of U has a neighborhood intersecting only finitely many of the sets V_m, and hence only finitely many of the corresponding collections $\mathcal{B}_m$. Thus $\mathcal{B}$ is the desired locally finite covering of U. □

The following is a well-known fact concerning the composition of smooth non-decreasing convex functions with convex or plurisubharmonic functions.

Lemma A.2. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a smooth convex function that is nondecreasing in each variable.*

1. *If $D \subset \mathbb{R}^d$ is a convex domain, and $u = (u_1, \ldots, u_N) : D \to \mathbb{R}^N$ is smooth, with each u_j convex on D, then $\Psi \circ u$ is convex on D.*
2. *If $\Omega \subset \mathbb{C}^d$ is a domain, and $u = (u_1, \ldots, u_N) : \Omega \to \mathbb{R}^N$ is smooth, with each u_j plurisubharmonic on Ω, then $\Psi \circ u$ is plurisubharmonic on Ω.*

Proof. We first prove the convex case. Since Ψ is smooth and nondecreasing in each variable, we have

$$\Psi_j \geq 0, \qquad j = 1, \ldots, N,$$

where $\Psi_j = \partial\Psi/\partial x_j$. The convexity of Ψ further gives its real Hessian

$$D^2\Psi \geq 0.$$

Let $U = \Psi \circ u$. By the chain rule,

$$D^2 U = \sum_{j=1}^{N} \Psi_j(u)\, D^2 u_j + \sum_{j,k=1}^{N} \Psi_{jk}(u)\, du_j \otimes du_k.$$

Thus, for every $\xi \in \mathbb{R}^d$,

$$D^2 U(\xi, \xi) = \sum_{j=1}^{N} \Psi_j(u)\, D^2 u_j(\xi, \xi) + \sum_{j,k=1}^{N} \Psi_{jk}(u)\, du_j(\xi)\, du_k(\xi).$$

The first sum is nonnegative because $\Psi_j \geq 0$ and each u_j is convex. The second sum is nonnegative because $D^2\Psi \geq 0$. Hence $D^2 U \geq 0$, and therefore $U = \Psi \circ u$ is convex.

The plurisubharmonic case is analogous. Let now $\Omega \subset \mathbb{C}^d$, and assume that each u_j is smooth plurisubharmonic. For every $\xi \in \mathbb{C}^d$, the complex Hessian of $U = \Psi \circ u$ is

$$H_U(\xi, \overline{\xi}) = \sum_{j=1}^{N} \Psi_j(u)\, H_{u_j}(\xi) + \sum_{j,k=1}^{N} \Psi_{jk}(u)\, \partial u_j(\xi)\, \overline{\partial u_k(\xi)}.$$

Since both terms are nonnegative by assumption,

$$H_U(\xi, \overline{\xi}) \geq 0$$

for every $\xi \in \mathbb{C}^d$. Thus $U = \Psi \circ u$ is plurisubharmonic. □

B A question on D'Angelo type of non-algebraic hypersurfaces

Determining whether a smooth hypersurface is of finite D'Angelo type is, in general, technically difficult in practice. In this section we present an elementary-looking (non-algebraic) real-analytic hypersurface in $\mathbb{C}^2$ for which a direct computation of the type does not appear to yield an effective method.

Question. Let $c_j \in \mathbb{C}$, $d_j \in \mathbb{R}$ and $a_j, r_j \in \mathbb{R}^+$, $j = 1, 2$, be constants, and let

$$U_j := \{(z, w) \in \mathbb{C}^2 : |z - c_j|^2 + |\operatorname{Im} w - d_j|^2 < r_j^2\}, \quad j = 1, 2$$

be two cylinders in $\mathbb{C}^2$. Set $U := U_1 \cap U_2$. Suppose that $U \neq \emptyset$ and consider the hypersurface M in U defined by

$$M = \{(z, w) \in U : \operatorname{Re} w = \sum_{j=1}^{2} a_j e^{\frac{1}{|z - c_j|^2 + |\operatorname{Im} w - d_j|^2 - r_j^2}}\}.$$

Is M of finite D'Angelo type at every point of M?

Note that the hypersurface M in the above is real-analytic. However, the global theorem of Diederich–Fornæss does not apply directly, since M is a locally defined real-analytic hypersurface and does not arise as a compact real-analytic variety. When only one cylinder is involved, however, finite type can be established by a direct argument, as the following example shows.

Example 5. *Let $c \in \mathbb{C}$, $d \in \mathbb{R}$, and $r \in \mathbb{R}^+$ be constants, and let*

$$U := \{(z, w) \in \mathbb{C}^2 : |z - c|^2 + |\operatorname{Im} w - d|^2 < r^2\}.$$

Consider the hypersurface M defined by

$$M = \{(z, w) \in U : \operatorname{Re} w = e^{\frac{1}{|z-c|^2+|\operatorname{Im} w - d|^2 - r^2}}\}.$$

Then M of finite D'Angelo type at every point of M.

Proof. The hypersurface M is real-analytic in U. Suppose, by contradiction, that M is of infinite D'Angelo type at some point $p \in M$. By D'Angelo's theorem, there exists a nonconstant holomorphic curve

$$\gamma(\zeta) = (z(\zeta), w(\zeta))$$

such that $\gamma(0) = p$ and $\gamma(\zeta) \in M$ for all ζ sufficiently close to 0.

Write $u = \operatorname{Re} w$ and $v = \operatorname{Im} w$. Along the curve γ, the defining equation of M gives

$$u(\zeta) = e^{\frac{1}{|z(\zeta)-c|^2+|v(\zeta)-d|^2-r^2}}.$$

Since $(z(\zeta), w(\zeta)) \in U$, we have $0 < u(\zeta) < 1$. Taking logarithms and then reciprocals, we obtain

$$\frac{1}{\ln u(\zeta)} = |z(\zeta) - c|^2 + |v(\zeta) - d|^2 - r^2.$$

Since z and w are holomorphic, while u and v are harmonic conjugates, applying the Laplacian with respect to ζ in the above gives

$$F''(u)|\nabla u|^2 = 4|z'|^2 + 2|\nabla v|^2,$$

where

$$F(s) := \frac{1}{\ln s}, \qquad 0 < s < 1.$$

Making use of the equalities $|\nabla u|^2 = |\nabla v|^2 = |w'|^2$, we further obtain

$$g(u(\zeta))|w'(\zeta)|^2 = 4|z'(\zeta)|^2, \tag{B.1}$$

where

$$g(s) := F''(s) - 2 = \frac{2 + \ln s}{s^2(\ln s)^3} - 2, \qquad 0 < s < 1. \tag{B.2}$$

We first consider the case $z' \equiv 0$. Then (B.1) implies

$$g(u(\zeta))|w'(\zeta)|^2 \equiv 0.$$

If $w' \equiv 0$, then γ is constant, a contradiction. Otherwise, on an open set where $w' \neq 0$, we have

$$g(u(\zeta)) \equiv 0.$$

Since $u = \operatorname{Re} w$ and w is holomorphic and nonconstant on this set, u is nonconstant there as well. Therefore, the image of u contains a nontrivial interval. Hence g must vanish identically on an interval. This is impossible from the explicit expression of g in (B.2).

It remains to consider the case $z' \not\equiv 0$. Then (B.1) also implies that $w' \not\equiv 0$. Hence there exists a point near which both z' and w' are nonzero. On a sufficiently small neighborhood, say V, of such a point, we may take logarithms in (B.1). This gives

$$\ln g(u(\zeta)) + \ln |w'(\zeta)|^2 = \ln 4 + \ln |z'(\zeta)|^2.$$

Since z' and w' are holomorphic and nonvanishing on this neighborhood, $\ln |z'|^2$ and $\ln |w'|^2$ are harmonic. Therefore,

$$\Delta(\ln g(u(\zeta))) = 0.$$

Noting that u is harmonic, this further yields

$$G''(u(\zeta))|\nabla u(\zeta)|^2 = 0,$$

where

$$G(s) := \ln g(s) = \ln \left(\frac{2 + \ln s}{s^2 (\ln s)^3} - 2 \right).$$

But $w' \neq 0$ on the chosen neighborhood, so $|\nabla u|^2 = |w'|^2 \neq 0$. Hence

$$G''(u(\zeta)) \equiv 0 \quad \text{on} \quad V.$$

Since u is nonconstant on V by assumption, its image contains an interval. Thus $G'' \equiv 0$ on an interval, and consequently

$$G(s) = as + b$$

there for some constants a, b. This contradicts the explicit expression of G, and thus completes the proof. $\square$

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